CS2209A 2017
Applied Logic for Computer Science

Lecture 3

Propositional Logic:

Equivalence and identities

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Revisit: truth table for $p \rightarrow q$

p	q	if p then q
True	*True*	True
True	*False*	**False**
False	*True*	**True**
False	*False*	**True**

- Let
  - p be "*It is raining*"
  - q be "*It is cloudy*"
  - **"If p then q"**
  - **"p implies q"**

- The implication is only **false** if its left hand side (i.e., ***p***) is true while the right hand side (***q***) is false.
- That is, "*if it is raining then it is cloudy*" is **false** only when it is raining out of blue sky. If it is not raining, this propositional formula is true no matter whether it is cloudy or not.

Logic with fun

- "If pigs can fly, then 2 + 2 = 4."
  True or False?

- "If pigs can fly, then 2 + 2 = 5."
  True or False?

Truth tables: equivalence

A	B	not A	if A then B	(not A) or B
True	*True*	False	True	True
True	*False*	False	False	False
False	*True*	True	True	True
False	*False*	True	True	True

- Now, $\neg \boldsymbol{A} \vee \boldsymbol{B}$ is the same as $\mathrm{A} \rightarrow \mathrm{B}$
  - So $\neg A \vee B$ and $\mathrm{A} \rightarrow \mathrm{B}$ are **equivalent**.

❖ "if it rains it must be cloudy" is equivalent to say "it can't happen that both it's not cloudy and raining".

Special types of sentences

- A sentence that has a satisfying assignment is **satisfiable.**
  – *Some* row in the truth table ends with *True*.
  – Example: $\mathbf{B \rightarrow A}$
- Sentence is a **contradiction:**
  – All assignments are falsifying.
  – *All* rows end with *False*.
  – Example: $\boldsymbol{A} \wedge \neg \boldsymbol{A}$
- Sentence is a **tautology:**
  – All assignments are satisfying
  – *All* rows end with *True*.
  – Example: $\mathbf{B \rightarrow A} \vee \boldsymbol{B}$

A	B	$\mathrm{B \rightarrow A}$
True	*True*	*True*
True	*False*	*True*
False	*True*	*False*
False	*False*	*True*

A	$\mathrm{A} \wedge \neg \mathrm{A}$
True	*False*
False	*False*

A	B	$\mathrm{A} \vee \boldsymbol{B}$	$\mathrm{B \rightarrow A} \vee \boldsymbol{B}$
True	*True*	*True*	*True*
True	*False*	*True*	*True*
False	*True*	*True*	*True*
False	*False*	*False*	*True*

Logical equivalence

- Two formulas F and G are **logically equivalent** ($F \Leftrightarrow G$ or $F \equiv G$) if they have the same value for every row in the truth table on their variables.
  - $\boldsymbol{A \wedge \neg A \equiv False}$ (same as saying it is a contradiction)
  - $\boldsymbol{(\neg A \vee B) \equiv (A \to B)}$
  - $\boldsymbol{(A \leftrightarrow B) \equiv (A \to B) \wedge (B \to A)}$
    - $\leftrightarrow$ is sometimes called the "bi-conditional"
    - $\leftrightarrow$ often pronounced as "if and only if", or "iff"

- Useful fact: proving that $F \equiv G$ can be done by proving that $F \leftrightarrow G$ is a **tautology**

Double negation

- **Negation cancels negation**
  - $\neg\neg A \equiv A$
  - “I do not disagree with you” = “I agree with you”

- For a human brain, harder to parse a sentence with multiple negations:
  - Alice says: “I refuse to vote against repealing the ban on smoking in public. “
    - Does Alice like smoking in public or hate it?

De Morgan’s Laws

- Simplifying negated formulas
  - For **AND**: $\neg\ (\boldsymbol{A} \wedge \boldsymbol{B})$ is equivalent to $(\neg \boldsymbol{A} \vee \neg \boldsymbol{B})$
  - For **OR**: $\neg\ (\boldsymbol{A} \vee \boldsymbol{B}) \equiv (\neg \boldsymbol{A} \wedge \neg \boldsymbol{B})$

- Example:
  - $\neg\ (\neg A \ \vee B)$ is $\neg\neg\ A \ \wedge \neg B$, same as $A \wedge \neg B$
  - So, since $(A \rightarrow B)$ is equivalent to $(\neg A \vee B)$,
    $\neg(A \rightarrow B)$ is equivalent to $A \wedge \neg B$

- Can be proved simply by truth tables

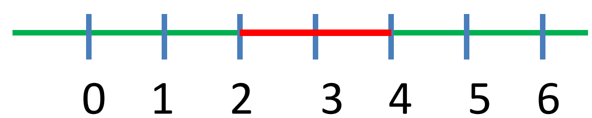
De Morgan's laws: examples

– Let A be "it's sunny" and B "it's cold".

- "It's sunny and cold today"! -- No, it's not!
- That could mean
  - No, it's not sunny.
  - No, it's not cold.
  - No, it's neither sunny nor cold.

- In all of these scenarios, "It's either not sunny or not cold" is true.

– Let A be "x < 2", B be "x > 4".



- "Either $x < 2$ or $x > 4$" – No, it is not!
- Then $2 \leq x \leq 4$

More examples

– Let A be “I play” and B “I win”.
  - $A \rightarrow B$: “If I play, then I win”
  - Equivalent to $\neg A \vee B$: “Either I do not play, or I win”.

– Negation: $\neg(A \rightarrow B)$: “It is not so that if I play then I win”.
  - By de Morgan’s law: $\neg(\neg A \vee B) \equiv (\neg\neg A \wedge \neg B)$
  - By double negation: $(\neg\neg A \wedge \neg B) \equiv (A \wedge \neg B)$
  - So negation of “If I play then I win” is “I play and I don’t win”.

More useful equivalences

- For any formulas A, B, C:
  - $A \vee \neg A \equiv True$ $\quad A \wedge \neg A \equiv False$
  - $True \vee A \equiv True.$ $\quad True \wedge A \equiv A$
  - $False \vee A \equiv A.$ $\quad False \wedge A \equiv False$
  - $A \vee A \equiv A \wedge A \equiv A$
- Also, like in arithmetic (with ∨ as +, ∧ as *)
  - $A \vee B \equiv B \vee A \quad and \quad (A \vee B) \vee C \equiv A \vee (B \vee C)$
  - Same holds for ∧.
  - Also, $(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C)$
- And unlike arithmetic
  - $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$

Logical identities

Name	$\wedge$-version	$\vee$-version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
	$p \wedge F \iff F$	$p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$

Longer example of negation

- Start with the outermost connective and keep applying **de Morgan's laws** and **double negation**. Stop when all negations are on variables.
- $\neg\,(\,(A \vee \neg B) \rightarrow \ (\neg A \wedge C)\,)$
  - $(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ (negating →)
  - $(A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ (de Morgan)
  - $(A \vee \neg B) \wedge (A \vee \neg C)$ (removing ¬¬)

– Can now simplify further, if we want to.
  - $A \vee (\neg B \wedge \neg C)$ (taking A outside the parentheses)

Simplifying formulas

- $\boldsymbol{A \land C \rightarrow (\neg B \lor C)}$
  - By $(\mathrm{F} \rightarrow G) \equiv (\neg F \lor G)$
    - equivalent to $\neg(A \land C) \lor (\neg B \lor C)$
  - De Morgan's law
    - $\neg(A \land C)$ is equivalent to $(\neg A \lor \neg C)$
  - So the whole formula becomes
    - $\neg A \lor \neg C \lor \neg B \lor C$
    - But $\neg C \lor C$ is always **true**!
    - So the whole formula is a tautology.