CS2209A 2017
Applied Logic for Computer Science

Lecture 4

Propositional Logic:

Simplifying formulas

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Review: Truth table

A	B	not A	A and B	A or B	if A then B
True	*True*	False	True	True	True
True	*False*	False	False	True	False
False	*True*	True	False	True	True
False	*False*	True	False	False	True

- “It is raining or I am a dolphin”
- "If pigs can fly, then 2 + 2 = 4."
  True or False?
- "If pigs can fly, then 2 + 2 = 5."
  True or False?

Review: Special types of sentences

A	B	$B \rightarrow A$
True	*True*	*True*
True	*False*	*True*
False	*True*	*False*
False	*False*	*True*

A	$A \wedge \neg A$
True	*False*
False	*False*

A	B	$A \vee B$	$B \rightarrow A \vee B$
True	*True*	*True*	*True*
True	*False*	*True*	*True*
False	*True*	*True*	*True*
False	*False*	*False*	*True*

- Which sentences are **satisfiable?**
  - $B \rightarrow A$, $A \vee B$, $B \rightarrow A \vee B$
- Which sentence is a **contradiction?**
  - $A \wedge \neg A$
- Which sentence is a **tautology?**
  - $B \rightarrow A \vee B$

Important tautologies

- **Law of the excluded middle** states that **($p \vee \neg p$) is a tautology**.
- In other words, p is either *true* or *false*, everything else is excluded.
- Proof: $p \vee \neg p$ is always *True*.

p	$\neg p$	$p \vee \neg p$
True	*False*	*True*
False	*True*	*True*

- Consider $(\neg(p \wedge q) \vee q)$. Is this formula a tautology? Give a proof for your answer.

Review: Logical equivalence

A	B	not A	if A then B	(not A) or B
True	*True*	False	True	True
True	*False*	False	False	False
False	*True*	True	True	True
False	*False*	True	True	True

- $\neg A \vee B$ and $\mathrm{A} \rightarrow \mathrm{B}$ are **equivalent**.

❖ Two formulas *F* and *G* are **logically equivalent** ($F \Leftrightarrow G$ or $F \equiv G$) if they have the same value for every row in the truth table on their variables.

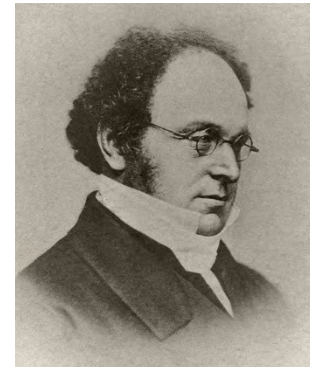
Review: Double negation

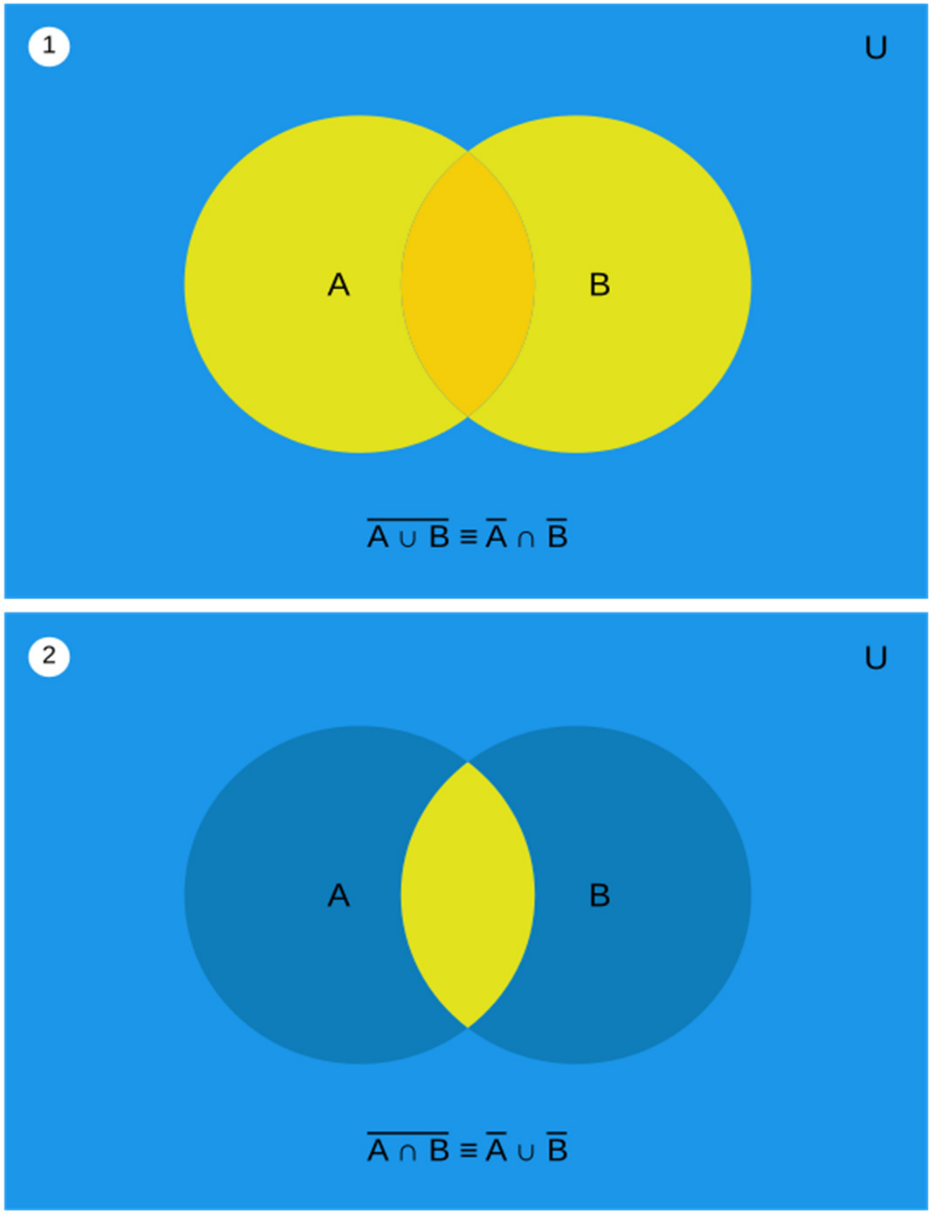
- **Double negation**
  - $\neg\neg\boldsymbol{A} \equiv \boldsymbol{A}$
  - “I do not disagree with you” = “I agree with you”
  - Negation cancels negation

- Review: De Morgan's Laws

  - For **OR**: $\neg (\boldsymbol{A} \vee \boldsymbol{B}) \equiv (\neg \boldsymbol{A} \wedge \neg \boldsymbol{B})$
  - For **AND**: $\neg (\boldsymbol{A} \wedge \boldsymbol{B}) \equiv (\neg \boldsymbol{A} \vee \neg \boldsymbol{B})$
  - The negation of a disjunction is the conjunction of the negations;
    the negation of a conjunction is the disjunction of the negations;
  - Useful for simplifying negated formulas

De Morgan's laws in set theory



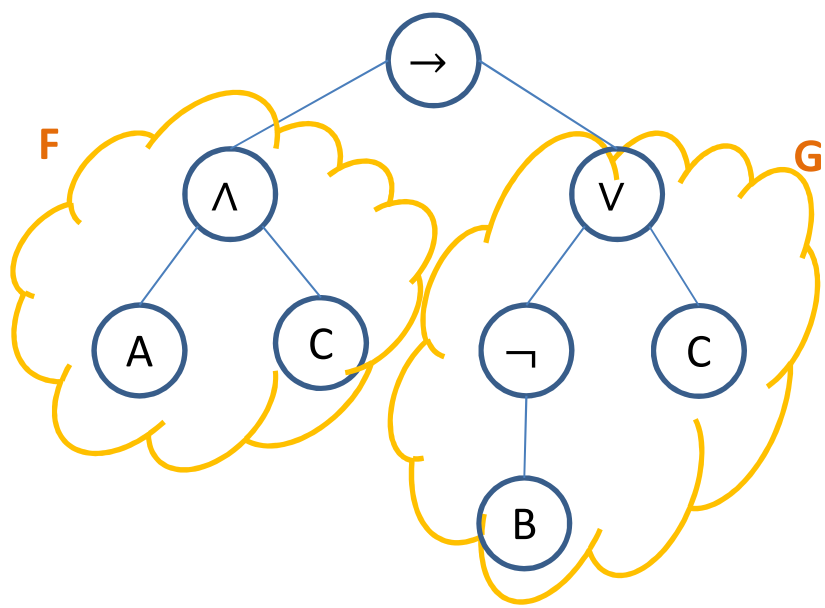


Simplifying formulas

- Start with the **outermost** connective and keep applying **de Morgan's laws** and **double negation**. Stop when all negations are on variables.
- **Precedence**: $\neg$ first, then $\wedge$, then $\vee$, $\rightarrow$ last
- **Example 1:** $\boldsymbol{A \wedge C \rightarrow (\neg B \vee C)}$
  - By $(\mathrm{F} \rightarrow G) \equiv (\neg F \vee G)$ (*let (A ∧ C) be F and $(\neg B \vee C)$ be G)
    - $\boldsymbol{A \wedge C \rightarrow (\neg B \vee C)} \equiv \neg(A \wedge C) \vee (\neg B \vee C)$
  - De Morgan's law
    - $\neg(A \wedge C)$ is equivalent to $(\neg A \vee \neg C)$
  - So the whole formula becomes
    - $\neg A \vee \neg C \vee \neg B \vee C$
    - $\equiv \neg A \vee \neg B \vee \neg C \vee C$ //commutativity
    - but $\neg C \vee C$ is always **true**! Now we get $\neg A \vee \neg B \vee$ *True*
    - So the whole formula is ***True***, a **tautology**.

Simplifying formulas

- $\boldsymbol{A \wedge C \rightarrow (\neg B \vee C)}$
  - **Order of precedence**: $\rightarrow$ **is the outermost**, that is, the formula is of the form $\boldsymbol{F \rightarrow G}$, where F is $(A \wedge C)$, and G is $(\neg B \vee C)$.



Simplifying formulas

- **Example 2:** $\neg\,(\,(\boldsymbol{A} \vee \neg \boldsymbol{B}) \rightarrow (\neg \boldsymbol{A} \wedge \boldsymbol{C})\,)$
  - $\equiv \neg\,(\,\neg(A \vee \neg B) \vee (\neg A \wedge C)\,)$ // →
  - $\equiv \neg\neg(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // de Morgan to ∨
  - $\equiv (A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // double negation
  - $\equiv (A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ //de Morgan to ∧
  - $\equiv (A \vee \neg B) \wedge (A \vee \neg C)$ //double negation

– Can now simplify further, if we want to.

  - $\equiv A \vee (\neg B \wedge \neg C)$ //distributivity, taking A outside the parentheses

Simplifying formulas

- **Example 3:** $(\boldsymbol{A} \wedge \neg \boldsymbol{B}) \rightarrow (\boldsymbol{A} \vee \boldsymbol{B} \rightarrow \neg \boldsymbol{B})$
  - $\equiv \neg(A \wedge \neg B) \vee (A \vee B \rightarrow \neg B)$ //$\rightarrow$
  - $\equiv \neg(A \wedge \neg B) \vee (\neg(A \vee B) \vee \neg B)$ //$\rightarrow$
  - $\equiv (\neg A \vee \neg\neg B) \vee (\neg(A \vee B) \vee \neg B)$ //De Morgan to $\wedge$
  - $\equiv (\neg A \vee B) \vee (\neg(A \vee B) \vee \neg B)$ //double negation
  - $\equiv \neg A \vee B \vee \neg B \vee (\neg(A \vee B))$ //associativity & commutativity
  - $\equiv \neg A \vee True \vee (\neg(A \vee B))$ //law of the excluded middle
  - $\equiv True$ //identity