CS2209A 2017
Applied Logic for Computer Science

Lecture 6

Propositional Logic:

Natural deduction

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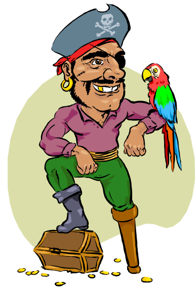
Treasure hunt

- In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humour and love of logical puzzles.

  In the note he wrote that he had hidden a treasure somewhere on the property.

  He listed 5 true statements and challenged the reader to use them to figure out the location of the treasure

Treasure hunt: statements



1. If this house is next to a lake, then a treasure is not in the kitchen
2. If the tree in the font yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. The tree in the front yard is an elm, or the treasure is buried under the flagpole
5. If the tree in the back yard is an oak, then the treasure is in the garage.

Treasure hunt

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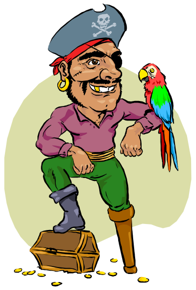
- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F

1. $A \rightarrow \neg B$
2. $C \rightarrow B$
3. A
4. $C \vee D$
5. $E \rightarrow F$

How many rows for a truth table ?

Natural deduction



- A: this house is next to a lake.
- B: the treasure is in the kitchen
- C: The tree in front is elm
- D: the treasure is under the flagpole.
- E: The tree in the back is oak
- F: The treasure is in the garage

1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F
6. **Not B**
7. **Not C**
8. **D**

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- **Therefore**, the treasure is not in the kitchen.

How do we get the intermediate steps?

Arguments

- An **argument**, in logic, is a sequence of propositional statements.
  - Called **argument form** when statements are formulas involving variables.

- The last statement in the sequence is called the **conclusion**. All the rest are **premises**.

$$
\begin{array}{c}
\boldsymbol{P_1} \\
\boldsymbol{P_2} \\
\vdots \\
\boldsymbol{P_n} \\
\hline
\therefore \boldsymbol{P_{n+1}}
\end{array}
$$

Treasure hunt

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5. If the tree in the back yard is an oak, then the treasure is in the garage.
6. The treasure is under the flagpole.

Premises

Conclusion

Argument

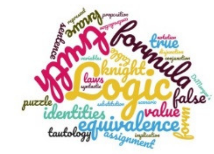
Arguments and validity

- An argument is **valid** if whenever all premises are true, the conclusion is also true.
  - So if premises are $P_1, \dots, P_n$, and conclusion is P_{n+1} ,
  - then the argument is valid

    if and only if
  - $\boldsymbol{P_1 \wedge P_2 \wedge \cdots P_n \rightarrow P_{n+1}}$ is a **tautology**

$$\begin{array}{c} \boldsymbol{P_1} \\ \boldsymbol{P_2} \\ \vdots \\ \boldsymbol{P_n} \\ \hline \therefore \boldsymbol{P_{n+1}} \end{array}$$

Valid vs Invalid argument



- **Valid argument**:

  AND of premises → conclusion is a **tautology**

$$\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \\ \hline \therefore P_{n+1} \end{array}$$

- If $x > 3$, then $x > 2$
- If $x > 2$, then $x \neq 1$
- $x > 3$

______________________ $\therefore$

$x \neq 1$

Valid

If $x > 3$, then $x > 2$

If $x > 2$, then $x \neq 1$

$x \neq 1$

______________________ $\therefore$

$x > 3$

Invalid

$(p \to q) \wedge (q \to r) \wedge r \to p$

is a not a tautology!

False when r is true, and p and q are both false.

Rules of inference

- Just like we used equivalences to simplify a formula instead of writing truth tables
- Can apply **tautologies** of the form $\mathbf{F} \rightarrow \boldsymbol{G}$
  - so that if F is an **AND** of several formulas derived so far, then we get G, and add G to the premises.
  - Such as $((\boldsymbol{p} \rightarrow \boldsymbol{q}) \wedge \boldsymbol{p}) \rightarrow \boldsymbol{q}$
- Keep going until we get the conclusion.

- If Socrates is a man, then
  Socrates is mortal
- Socrates is a man

________________________ ∴

Socrates is mortal

Modus ponens

- The main **rule of inference**, given by the **tautology** $(p \to q) \land p \to q$, is called **Modus Ponens** ("method of affirming" in Latin).

> - **If** p **then** q
> - p
>
> ______________ $\therefore$
>
> q

p	q	$p \to q$	$(p \to q) \land p$	$(p \to q) \land p \to q$
True	*True*	True	True	True
True	*False*	False	False	True
False	*True*	True	False	True
False	*False*	True	False	True

Modus ponens: treasure hunt

- **If** $\boldsymbol{p}$ **then** $\boldsymbol{q}$
- $\boldsymbol{p}$

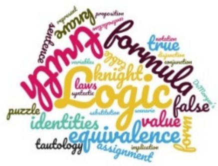
$\overline{\hphantom{xxxxxxxxxx}}$ $\therefore$

$\boldsymbol{q}$

- If house is next to the lake then the treasure is not in the kitchen
- The house is next to the lake
- Therefore, the treasure is not in the kitchen.

➢ **Here, p is “the house is next to the lake”, and q is “the treasure is not in the kitchen”.**

Hypothetical syllogism



- If p then q
- If q then r

______________________ $\therefore$

If p then r

- If this house is next to a lake, then a treasure is not in the kitchen
- If the treasure is not in the kitchen, then the tree in the font yard is not an elm.

____ $\therefore$

If this house is next to a lake, the tree in the front yard is not an elm.

Disjunctive syllogism

- p **or** q
- **Not** p

______________ ∴

q

- p **or** q
- **Not** q

______________ ∴

p

- It is either day or night
- It is not night

______________________ ∴

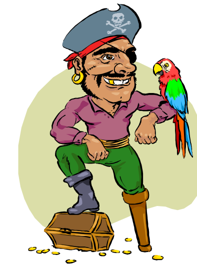
It is day

- The tree in the front yard is an elm, or the treasure is buried under the flagpole
- The tree in the front yard is not an elm

________________________________ ∴

The treasure is buried under the flagpole

Natural deduction for treasure hunt



1. If A then not B
2. If C then B
3. A
4. C or D
5. If E then F
6. Not B
7. If not B then not C
8. Not C
9. **D**

➢ D: the treasure is under the flagpole.

- If A then $not\ B$
- $\boldsymbol{A}$

$\overline{}\ \therefore$

$\boldsymbol{Not\ B}$

“If C then B” is equivalent to “If not B then not C”
Contrapositive !

- If not B then $not\ C$
- $\boldsymbol{Not\ B}$

$\overline{}\ \therefore$

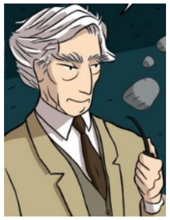
$\boldsymbol{Not\ C}$

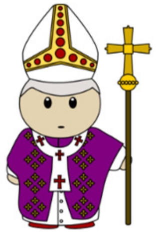
- $\boldsymbol{C}$ or D
- $\boldsymbol{Not\ C}$

$\overline{}\ \therefore$

$\boldsymbol{D}$

False premises





- An argument can still be valid when some of its premises are false.
  - Remember, false implies anything.
- **Bertrand Russell**:
  “If 2+2=5, then I am the pope”

Puzzle: can you see how to prove this?