Tutorial #4

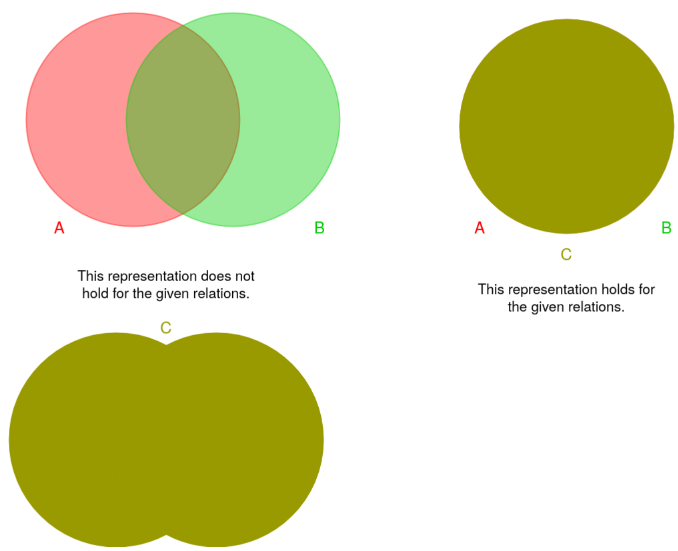
Problem 1 Show that if the sets A, B and C satisfy the following relations simultaneously:

1. $A \cup B \; = \; C$,
2. $(A \cup C) \cap B \; = \; C$,
3. $(A \cap C) \cup B \; = \; A$

then they are equal.

Solution 1

We propose two solutions. The first one is short, but requires some intuition and practice, which can be gained by drawing pictures, like the one below. The second solution is longer and less elegant; but it is also more detailed.



Short solution. There are essentially four steps.

1. From $A \cup B \; = \; C$, we derive $A \subseteq C$ and $B \subseteq C$.
2. Thus $A \cup C$ becomes C and $(A \cup C) \cap B \; = \; C$ becomes $C \cap B \; = \; C$, which means $C \subseteq B$
3. Similarly, $A \cap C$ becomes A and $(A \cap C) \cup B \; = \; A$ becomes $A \cup B \; = \; A$, which means $B \subseteq A$.

4. Finally, we have
$$B \subseteq A \subseteq C \subseteq B,$$
which implies $B = A = C$.

Longer and detailed solution. The first hypothesis, namely $A \cup B = C$ implies that both A and B are contained in C, that is
$$A \subseteq C \text{ and } B \subseteq C.$$
Now looking at the third hypothesis, namely $(A \cap C) \cup B = A$, we see that by applying distributivity of $\cup$ over $\cap$ we obtain:
$$(A \cup B) \cap (C \cup B) = A.$$
Using the first hypothesis, we deduce:
$$C \cap (C \cup B) = A.$$
Now using distributivity of $\cap$ over $\cup$, we further deduce:
$$C \cup (C \cap B) = A.$$
Since $C \cap B$ is contained in C, we have $C \cup (C \cap B) = C$ and we derive this equality:
$$C = A.$$
It remains to be proved that $A = B$ holds. Using $C = A$ within the the second hypothesis, namely $(A \cup C) \cap B = C$, we obtain:
$$(A \cup A) \cap B = A,$$
that is,
$$A \cap B = A.$$
This latter equality implies that every element of A is in the intersection of A and B, thus, every element of A is in B. In other words, we have $A \subseteq B$. Now, remember that we saw $B \subseteq C$ and $C = A$, which implies $B \subseteq A$. Finally, we have $B = A$ and the three sets are indeed equal.

Problem 2 Prove the following set identities:
1. $(B \setminus A) \cup (C \setminus A) = (B \cup C) \setminus A$,
2. $(A \setminus B) \setminus (B \setminus C) = A \setminus B$.

Solution 2 Recall that $B \setminus A$ means $B \cap \overline{A}$.

1. We prove that $(B \setminus A) \cup (C \setminus A)$ can be rewritten as $(B \cup C) \setminus A$. First, we use the fact $B \setminus A = B \cap \overline{A}$ and $C \setminus A = C \cap \overline{A}$ hold. Hence, we have:
$$(B \setminus A) \cup (C \setminus A) \; = \; (B \cap \overline{A}) \cup (C \cap \overline{A})$$
Next, we apply distributivity of $\cup$ over $\cap$, leading to:
$$(B \setminus A) \cup (C \setminus A) \; = \; (B \cup C) \cap \overline{A},$$
from which we finally derive
$$(B \setminus A) \cup (C \setminus A) \; = \; (B \cup C) \setminus A.$$

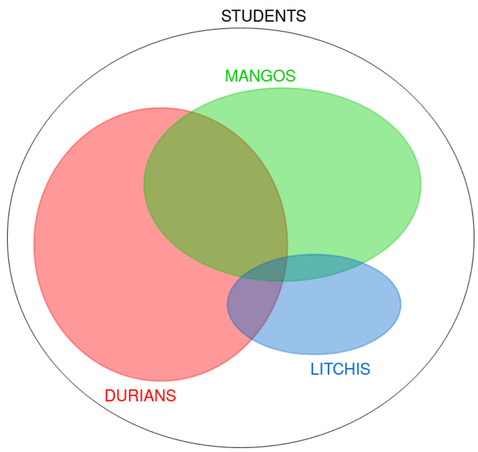
2. We have the following chain of equalities:
$$\begin{aligned}
(A \setminus B) \setminus (B \setminus C) &= (A \cap \overline{B}) \cap \overline{B \cap \overline{C}} \\
&= (A \cap \overline{B}) \cap (\overline{B} \cup C) \\
&= (A \cap \overline{B} \cap \overline{B}) \cup (A \cap \overline{B} \cap C) \\
&= (A \cap \overline{B}) \cup (A \cap \overline{B} \cap C) \\
&= A \cap \overline{B} \\
&= A \setminus B.
\end{aligned}$$

Problem 3 In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Solution 3 Denote by D, M, L the sets of students who have eaten durians, mangoes and litchis. The key observation is that the following 8 sets

$$D \cap M \cap L, D \cap M \cap \overline{L}, D \cap \overline{M} \cap L, D \cap \overline{M} \cap \overline{L}, \overline{D} \cap M \cap L, \overline{D} \cap M \cap \overline{L}, \overline{D} \cap \overline{M} \cap L, \overline{D} \cap \overline{M} \cap \overline{L}$$

are pairwise disjoint. In fact, they form what is called a *partition* of the whole set of students. This is illustrated by the picture below, where the sets D, M, L are respectively red, green and blue. We see, on that picture, that there are 8 regions corresponding respectively to those 8 sets.



The strategy for solving this type of problem is to compute one after another the cardinality of each set in the partition. We start from the "center" of the picture, that is, from $D \cap M \cap L$, of which we know the cardinality. Then, we compute the cardinalities of sets corresponding to "have eaten exactly two fruits", that is, $D \cap M \cap \overline{L}$, $D \cap \overline{M} \cap L$ and $\overline{D} \cap M \cap L$. Indeed, we can easily deduce the cardinalities of those sets from the hypotheses of the problem. Then, we proceed with the sets corresponding to "have eaten exactly one fruit", that is, $D \cap \overline{M} \cap \overline{L}$, $\overline{D} \cap M \cap \overline{L}$ and $\overline{D} \cap \overline{M} \cap L$. Finally, we can deduce the cardinality of the set "have eaten no fruits", that is, $\overline{D} \cap \overline{M} \cap \overline{L}$.

From the hypotheses and the above strategy, we have

1. $|D \cap M \cap L| = 5$,
2. $|D \cap M \cap \overline{L}| = 34 - 5 = 29$,
3. $|D \cap \overline{M} \cap L| = 16 - 5 = 11$,
4. $|\overline{D} \cap M \cap L| = 12 - 5 = 7$,
5. $|D \cap \overline{M} \cap \overline{L}| = 88 - (29 + 11 + 5) = 43$,
6. $|\overline{D} \cap M \cap \overline{L}| = 73 - (7 + 29 + 5) = 32$,
7. $|\overline{D} \cap \overline{M} \cap L| = 46 - (11 + 7 + 5) = 23$,
8. $|\overline{D} \cap \overline{M} \cap \overline{L}| = 200 - (5 + 29 + 11 + 43 + 7 + 32 + 23) = 50$.

Problem 4 Prove that for the sets A, B, C, D, we have:

$$(A \times B) \cup (C \times D) \ \subseteq \ (A \cup C) \times (B \cup D).$$

Does the equality hold?

Solution 4 Proving that the union $X \cup Y$ of two sets X, Y is contained in a third set Z is equivalent to prove that each set X and Y is contained in Z. We used that remark here. Therefore, in this problem, it is sufficient to prove that

1. $(A \times B) \subseteq (A \cup C) \times (B \cup D)$,
2. $(C \times D) \subseteq (A \cup C) \times (B \cup D)$,

both hold. Next, we observe that the pair (A, B) and the pair (C, D) play a symmetric role, that is, if we exchange A with C, and B with D, then the former inclusion becomes the latter inclusion. In other words, it is sufficient to prove that one of these two inclusions holds. So, let us prove the former one, that is:

$$(A \times B) \subseteq (A \cup C) \times (B \cup D).$$

Let a be in A and b be in B, that is, Let (a, b) be in $(A \times B)$. We want to prove that $(a, b) \in (A \cup C) \times (B \cup D)$ holds as well. Since $a \in A$ holds, we also have $a \in A \cup C$. Similarly, since $b \in B$ holds, we also have $b \in B \cup D$. Hence, we have $(a, b) \in (A \cup C) \times (B \cup D)$. Since the element (a, b) is an arbitrary element of $(A \times B)$, we have proved the desired inclusion. Similarly, with the "symmetry" argument, the other inclusion, namely $(C \times D) \subseteq (A \cup C) \times (B \cup D)$, holds as well and the proof is complete.

Remains to answer the question whether the equality

$$(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D).$$

holds or not?

Thinking about the analogy of union and Cartesian product of sets with addition and multiplication of numbers, respectively, we can guess that the answer is no. Let us try an example which looks "general enough" say, with $A = \{a\}$, $B = \{b\}$, $C = \{c\}$ and $D = \{d\}$. Then, we have:

1. $(A \times B) \cup (C \times D) = \{(a, b), (c, d)\}$,
2. $(A \cup C) \times (B \cup D) = \{(a, b), (a, d), (c, b), (c, d)\}$.

Clearly, the inclusion does not hold in this case. In fact, for the inclusion to hold we would have either $A = C = \emptyset$ or $B = D = \emptyset$.