The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

UWO – September 26, 2021

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Plan for Part III

1. Basic Proof Methods

1.1 Mathematical Statements and their proofs
1.2 Proving Conditional Statements
1.3 Theorems that are Biconditional Statements
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Plan for Part III

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   d. showing that system specifications are consistent, etc.

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5. A *conjecture* is a statement that is being proposed to be true. Once a proof of a conjecture is found, it becomes a theorem. It may turn out to be false.

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   a. For example, the statement:

      "If $x > y$ holds, where x and y are positive real numbers, then $x^2 > y^2$ holds as well"

   b. really means:

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3. By universal generalization (UG) (an inference rule, opposite of universal instantiation UI) the truth of the original formula follows.
4. So, we must prove something of the form:
$$p \rightarrow q$$

Plan for Part III

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3. Even though these examples seem silly, both trivial and vacuous proofs are often used in mathematical induction, as we will see in Chapter 5.

Even and odd integers

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2. We will need this basic fact about the integers in some of the example proofs to follow.

Proving conditional statements: $p \rightarrow q$: direct proof

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The symbol ■ marks the end of the proof and is referred to as a 'tombstone.' Sometimes **QED** (abbreviation for the Latin sentence "quod erat demonstrandum", meaning "what was to be demonstrated") or $\lhd$ is used instead.

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   a. Assume n is even.
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   c. Thus, we have $3n+2 = 3(2k)+2 = 6k+2 = 2(3k+1) = 2j$ for $j = 3k+1$.

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   d. Therefore, we have proved that $3n + 2$ is even.
   e. Since we have shown $\neg q \to \neg p$, then $p \to q$ must hold as well.
   f. If n is an integer and $3n + 2$ is odd (not even), then n is odd (not even). ■

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    e. We have shown that if n is an even integer, then n^2 is even. Therefore by contraposition, if n^2 is odd, then n is odd. ∎

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    g. Therefore, there is no largest prime. ■

Plan for Part III

Theorems that are biconditional statements

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Sometimes **iff** is used as an abbreviation for "**if an only if** ," as in "If n is an integer, then n is odd iif n^2 is odd."

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Solution: Step 5. $a - b = 0$ by the premise and division by 0 is undefined.

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Plan for Part III

1. Basic Proof Methods
   1.1 Mathematical Statements and their proofs
   1.2 Proving Conditional Statements
   1.3 Theorems that are Biconditional Statements
   1.4 Errors in proofs

2. Proof Strategies
   2.1 Proof by case inspection
   2.2 Without Loss of Generality
   2.3 Existence Proofs
   2.4 Counterexamples
   2.5 Uniqueness Proofs
   2.6 Proof Strategies for implications
   2.7 Backward Reasoning
   2.8 Universally Quantified Assertions
   2.9 Open Problems
   2.10 Additional proof methods

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3. Therefore, one can prove each of the implications (**cases**) of $p_i \to q$ <u>separately</u>.

Proof by case inspection: example

1. Define $a \,@\, b \equiv max a, b$. That is:

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$$(a \mathbin{@} b) \mathbin{@} c = a \mathbin{@} (b \mathbin{@} c)$$

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4. **Proof**: Let a, b, and c be arbitrary real numbers. Then one of the following 6 cases must hold:

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$$(p_1 \lor p_2 \lor p_3 \lor p_4 \lor p_5 \lor p_6) \rightarrow (a\,@b)\,@\,c = a\,@\,(b\,@\,c)$$

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   b. $(a@b) = a$, $a@c = a$, $b@c = b$
   c. Hence $(a@b)@c = a = a@(b@c)$
   d. Therefore the equality holds for the first case.
2. A complete proof requires that the equality be shown to hold for all 6 cases. But the proofs of the remaining cases are similar. Try them. ■

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5. Therefore, for any integer y, the integers $x \cdot y$ *and* $x + y$ are not both even. ∎
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7. The phrase *without loss of generality* (WLOG) indicates this.

Plan for Part III

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2. **Constructive** existence proof:

Godfrey Harold Hardy (1877-1947)

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Note, at the end of this proof we know that x^y is rational either for x=y= $\sqrt{2}$ or for $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$ (exclusive or) but we do not know for which specific pair.

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Example: "Every positive integer is the sum of the squares of 3 integers." The integer 7 is a counterexample. So the claim is false.

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   - b. *Uniqueness*: Suppose that there is also a real number s such that $as + b = 0$. Then $ar + b = as + b$, where $r = -\frac{b}{a}$. Subtracting b from both sides and dividing by a shows that $r = s$. ■

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Plan for Part III

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4. Now reasoning forward, the first player can ensure a win by removing 3 stones and leaving 12.

Plan for Part III

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Case 2 on the next slide.

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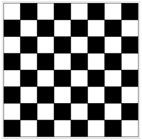
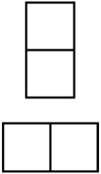
3. Therefore we have shown that x is even if and only if x^2 is even. ∎

Proof and disproof: Tilings

Example 1: Can we tile the standard checker-board using dominos?



Standard Checkerboard



Two Dominoes

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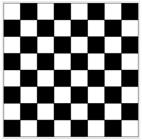
Solution: Yes! One example provides a constructive existence proof.



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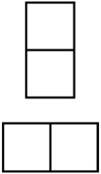

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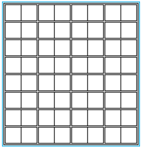

One Possible Solution

Example 2: Can we tile a checker-board obtained by removing one of the four corner squares of a standard checker-board?

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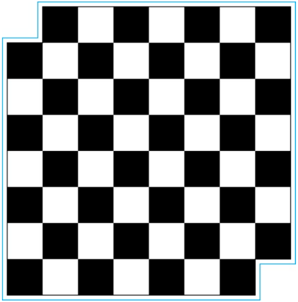
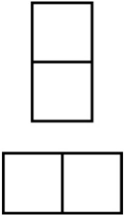
c. The number 63 is not even.
d. We have a contradiction. ∎

Tilings

Example 3: Can we tile a board obtained by removing both the upper left and the lower right squares of a standard checker-board?



Nonstandard Checker-board



Two Dominoes

Continued on next slide

Tilings

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f. Contradiction! ∎

Plan for Part III

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   A proof was found by Andrew Wiles in the 1990s.

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1. **The 3x + 1 Conjecture**: Let T be the transformation that sends an even integer x to $\frac{x}{2}$ and an odd integer x to $3x + 1$. For all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1.

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An open problem

1. **The 3x + 1 Conjecture**: Let T be the transformation that sends an even integer x to $\frac{x}{2}$ and an odd integer x to $3x + 1$. For all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1.

   For example, starting with $x = 13$:

   $\mathrm{T}(13) = 3{\cdot}13 + 1 = 40$, $\mathrm{T}(40) = 40/2 = 20$, $\mathrm{T}(20) = 20/2$ $= 10$,

   $\mathrm{T}(10) = 10/2 = 5$, $\mathrm{T}(5) = 3{\cdot}5 + 1 = 16$,$\mathrm{T}(16) = 16/2 = 8$,

   $\mathrm{T}(8) = 8/2 = 4$, $\mathrm{T}(4) = 4/2 = 2$, $\mathrm{T}(2) = 2/2 = 1$

   The conjecture has been verified using computers up to $5 \times 6 \times 10^{13}$.

Plan for Part III

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   c. Cantor diagonalization is used to prove results about the size of infinite sets.
   d. Combinatorial proofs use counting arguments.