Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

UWO – October 3, 2021

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2. Set theory is an important branch of mathematics:
   1. Many different systems of axioms have been used to develop set theory.
   2. Here, we are not concerned with a formal set of axioms for set theory.
   3. Instead, we will use what is called naïve set theory.

Plan for Part I

1. Sets

1.1 Defining sets
1.2 Venn Diagram
1.3 Set Equality
1.4 Subsets
1.5 Venn Diagrams and Truth Sets
1.6 Set Cardinality
1.7 Power Sets
1.8 Cartesian Products

2. Set Operations

2.1 Boolean Algebra
2.2 Union
2.3 Intersection
2.4 Complement
2.5 Difference
2.6 The Cardinality of the Union of Two Sets
2.7 Set Identities
2.8 Generalized Unions and Intersections

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4. If a is not a member of A, write $a \notin A$

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$\mathbb{C}$ = set of *complex numbers*.

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3. **Example**: The truth set of $P(x)$ where the domain is the integers and $P(x)$ is "$|x| = 1$" is the set {-1,1}

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   b. **Important**: the empty set is different from a set containing the empty set:

$$\varnothing \neq \{ \varnothing \}$$

Plan for Part I

Venn Diagram

John Venn (1834 - 1923) Cambridge, UK

Sets and their elements can be represented via Venn diagrams

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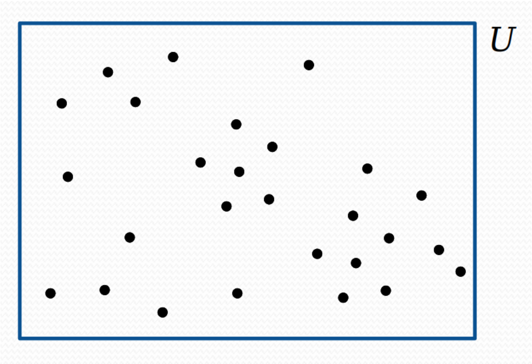

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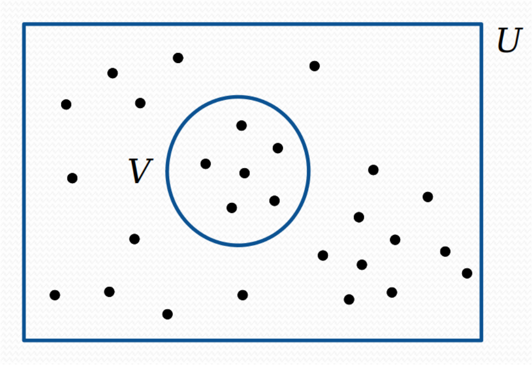

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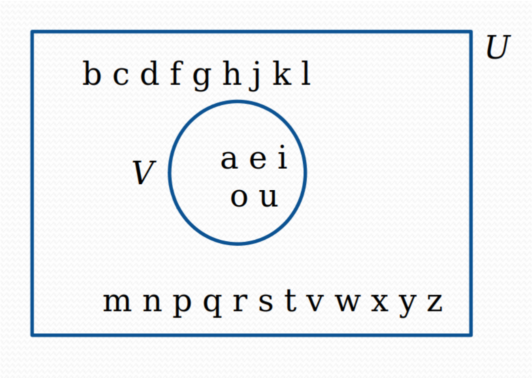

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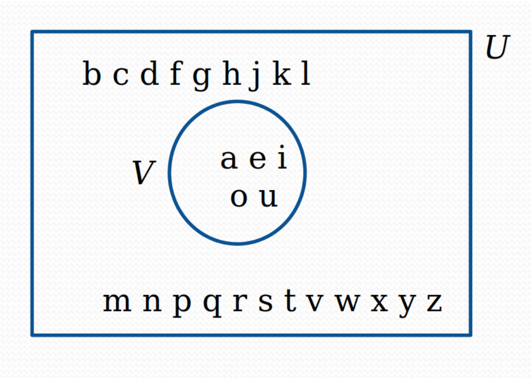

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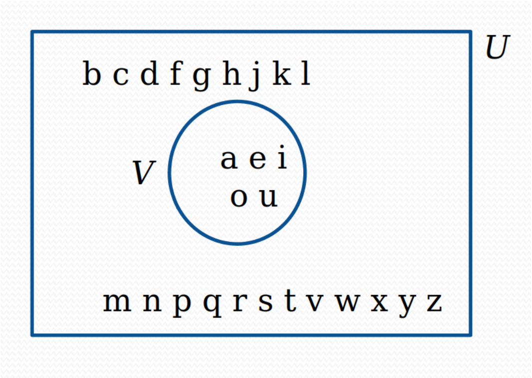

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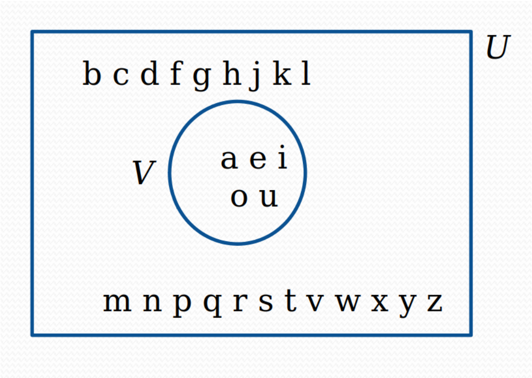

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◯ – Set *V*: all vowels

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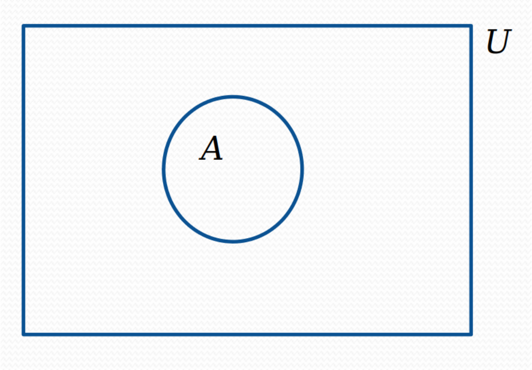

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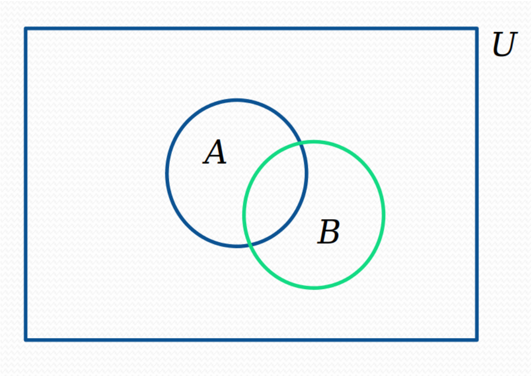

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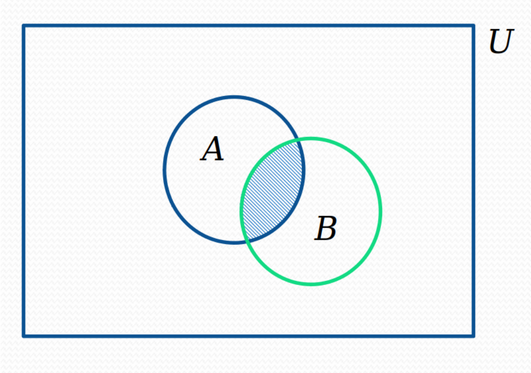

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2. *Example*: shaded area illustrates a set of elements that are in both sets A and B (i.e. *intersection* of two sets, see later). E.g consider A={a,b,c,f,z} and B={c,d,e,f,x,y}.

Plan for Part I

Set equality

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$$\{1, 3, 5\} = \{3, 5, 1\}$$
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2. $A \subseteq B$ holds if and only if $\forall x \ (x \in A \rightarrow x \in B)$ is true.
3. Observe that:
   a. Because $a \in \varnothing$ is always false, for every set S we have:
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   $$\emptyset \subseteq S.$$
   b. Because $a \in S \rightarrow a \in S$, for every set S, we have:
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Showing that a set is or is not a subset of another set

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   1. The set of all computer science majors at your school is a subset of all students at your school.
   2. The set of integers with squares less than 100 is not a subset of the set of all non-negative integers.

Another look at equality of sets

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3. This is also equivalent to:

$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

Proper subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then the following is true:

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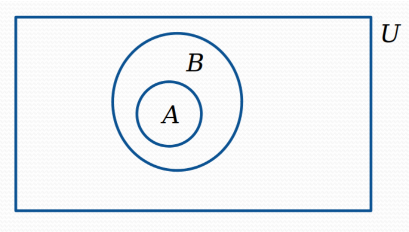
Venn Diagram for a proper subset $A \subset B$



Example: $A = \{c, f, z\}$ and $B = \{a, b, c, d, e, f, t, x, z\}$.

Plan for Part I

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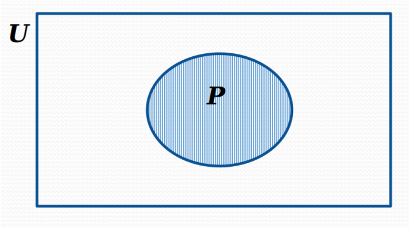

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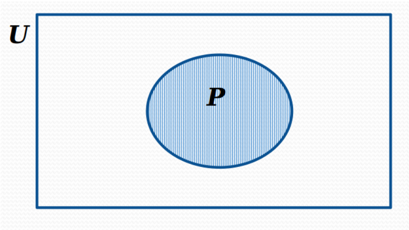

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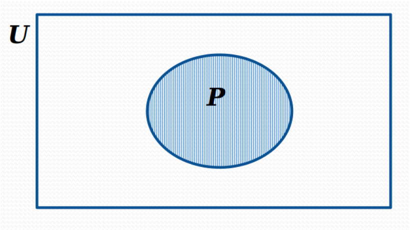

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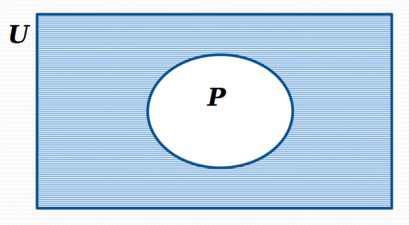

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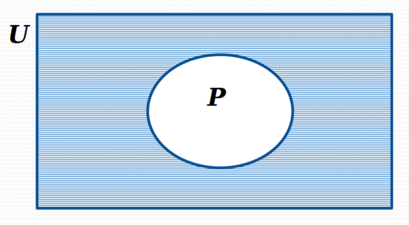

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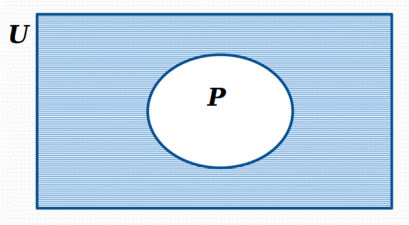

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Note that: $x \notin P \equiv \neg P(x)$

Venn Diagrams and logical connectives: conjunctions

Consider two arbitrary predicates $P(x)$ and $Q(x)$ defined for elements x in U together with their corresponding truth sets $P = \{x \mid P(x)\}$ and $Q = \{x \mid Q(x)\}$.

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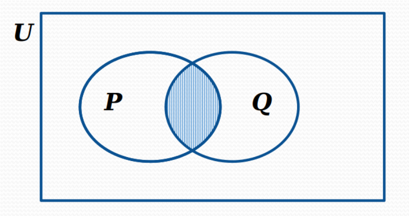

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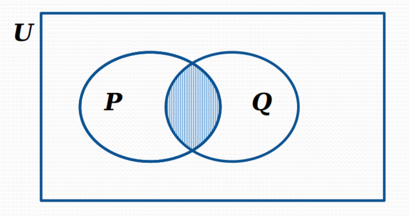

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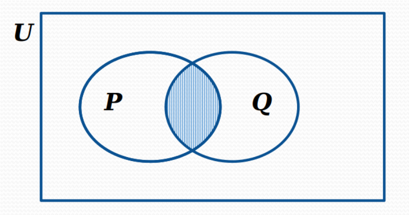

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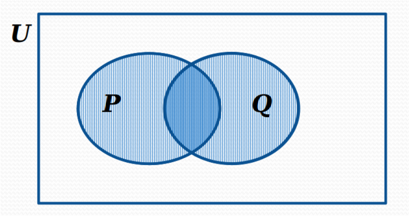

 – truth set $\{x \mid P(x) \vee Q(x)\}$ that is, all elements x where $P(x)$ or $Q(x)$ is true

Same as *union* of sets P and Q (see section 2.2)

Venn diagrams and logical connectives: implications

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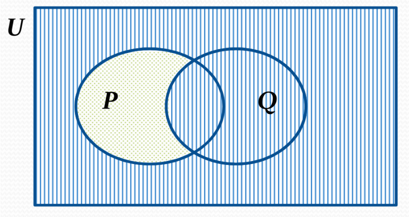

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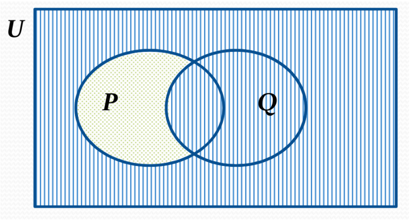

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$$p \rightarrow q \equiv \neg p \vee q$$

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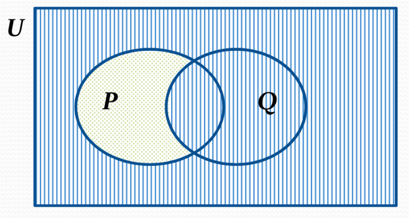

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2. Thus, we have:

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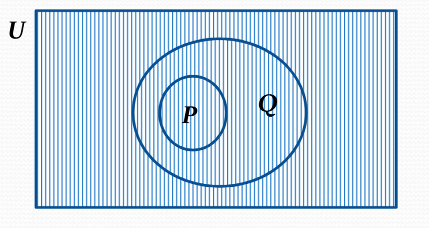

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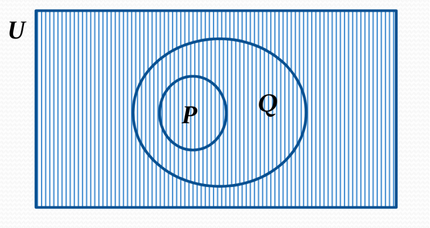

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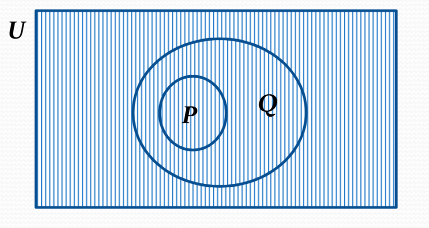

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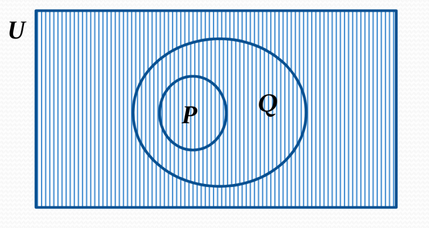

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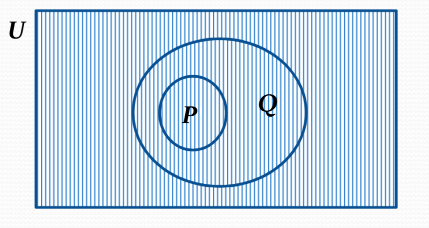

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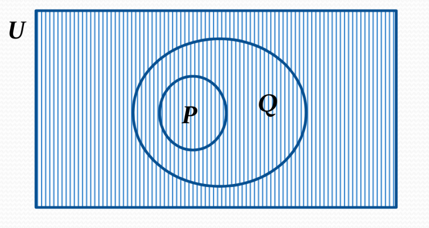

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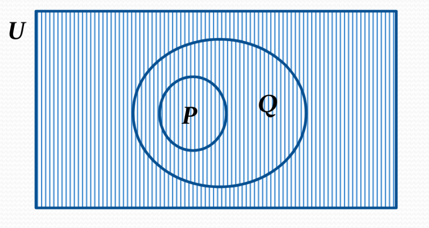

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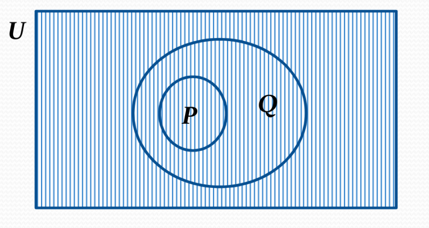

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Venn diagram and implication: special case

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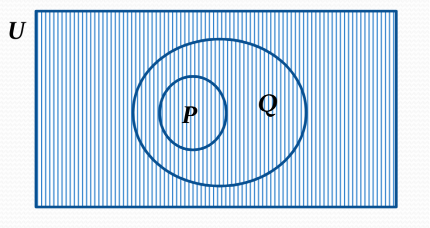

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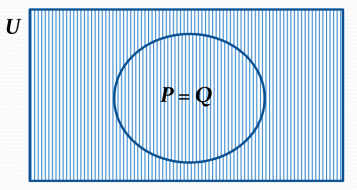

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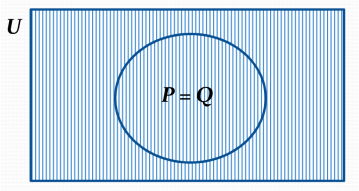

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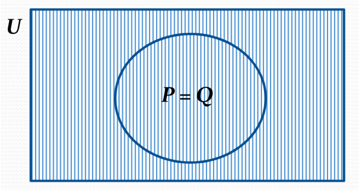

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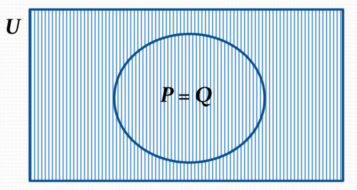

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2. In Chapters 5 and 6, we will discuss different ways to show this.

Plan for Part I

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4. The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

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(1596-1650)

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

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Solution: $A \times B \times$ C $=$ {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Plan for Part I

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4. All sets $A, B, \ldots$ shown in the next slides are assumed to be subsets of U.

Plan for Part I

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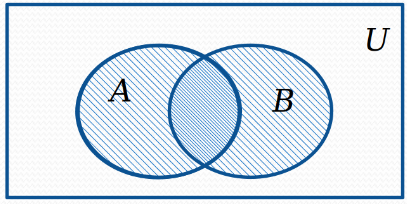

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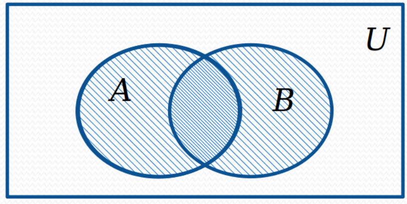

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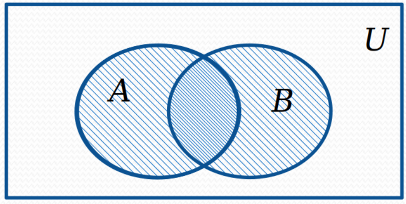

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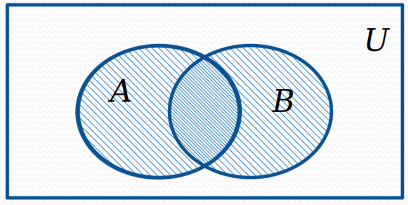

Union is analogous to disjunction, see earlier slides.

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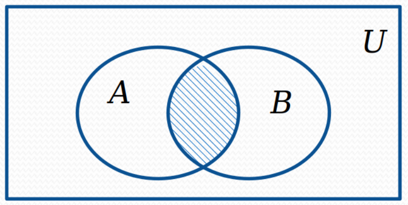

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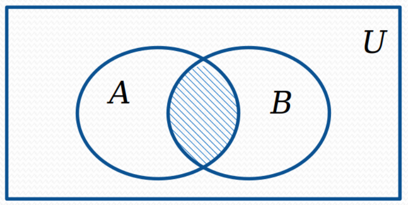

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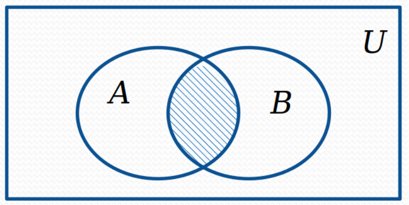

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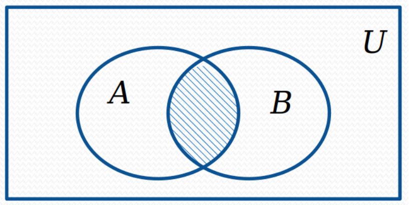

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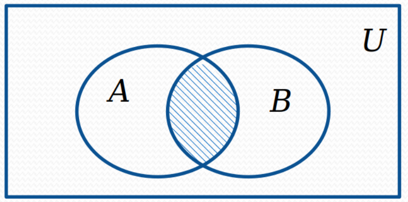

Intersection is analogous to conjunction, see earlier slides.

Plan for Part I

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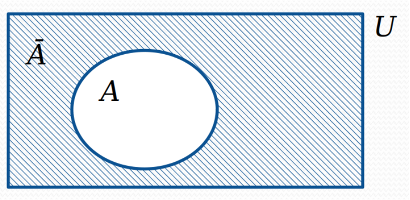

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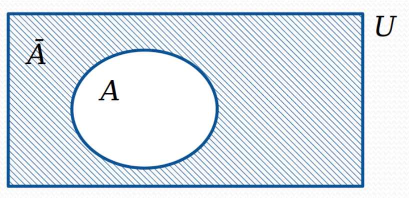

Complement is analogous to negation, see earlier.

Plan for Part I

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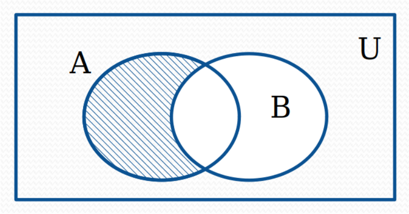
Note: $\overline{A} = U - A$

Venn Diagram for $A - B$

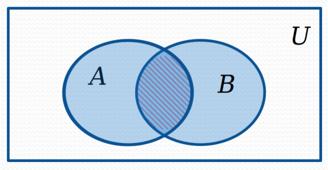


Plan for Part I

The cardinality of the union of two sets

1. Inclusion-Exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

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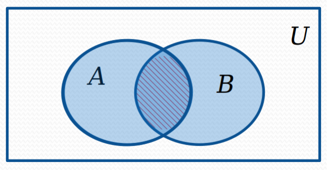

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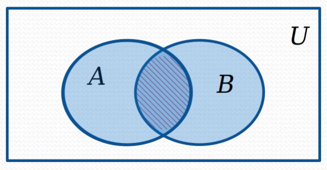

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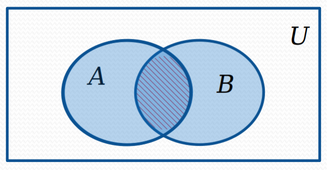

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   a. Let A be the math majors in your class and B be the CS majors in your class.
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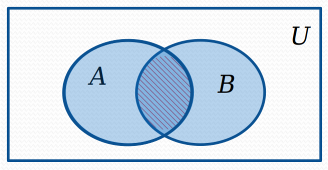
The cardinality of the union of two sets



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   b. To count the number of students in your class who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
   c. We will return to this principle in Chapter 6 and Chapter 8, where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Review questions

Example: Given $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$ solve the following:

1. $A \cup B$

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   Solution: $\{1,2,3\}$
6. $B - A$

   Solution: $\{6,7,8\}$

Plan for Part I

Set identities

1. Identity laws

$$A \cup \varnothing = A \qquad A \cap U = A$$

Set identities

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4. Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide ↪

Set identities

1. Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

Set identities

1. Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

2. Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

Set identities

1. Commutative laws

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2. Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C \qquad A \cap (B \cap C) = (A \cap B) \cap C$$

3. Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide ↪

Set identities

1. De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

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$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

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$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

3. Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$$

Proving set identities

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   c. Membership tables

(to be explained)

Proof of second De Morgan law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof of second De Morgan law

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Solution: We prove this identity by showing that:

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2. $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide ↪

Proof of second De Morgan law

These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

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$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	definition of complement

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$(x \in \overline{A}) \vee (x \in \overline{B})$	definition of complement
$x \in \overline{A} \cup \overline{B}$	definition of union

Continued on next slide ↪

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$\neg x \in (A \cap B)$	definition of intersection

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$\neg((x \in A) \wedge (x \in B))$	De Morgan's 1^{st} Law
$\neg x \in (A \cap B)$	definition of intersection
$x \in \overline{A \cap B}$	definition of complement

$\blacksquare$

Set-builder notation: second De Morgan law

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ by definition of complement

Set-builder notation: second De Morgan law

$\overline{A \cap B}$	$= \{x \mid x \notin A \cap B\}$	by definition of complement
	$= \{x \mid \neg x \in (A \cap B)\}$	by definition of ‘not in’

Set-builder notation: second De Morgan law

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\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg x \in (A \cap B)\} && \text{by definition of `not in'} \\
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\end{aligned}
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&= \overline{A} \cup \overline{B} && \text{by definition of notation}
\end{aligned}
$$

∎

Membership table

Example: Construct a membership table to show that the distributive law holds:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
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1	1	0					
1	0	1					
1	0	0					
0	1	1					
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1	1	1	1	1			
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
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A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1		
1	1	0					
1	0	1					
1	0	0					
0	1	1					
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1	1	0	0				
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1	0	1	0	1	1	1	1
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0	1	1	1	1	1	1	1
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0	1	1	1	1	1	1	1
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0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Plan for Part I

Generalized unions and intersections

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