Number Theory and Cryptography

Chapter 4: Part I

UWO – October 20, 2021

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4. Number theory has long been studied because of the beauty of its ideas, its accessibility, and its wealth of open questions.
5. We will use many ideas developed in Chapter 1 about proof methods and proof strategies in our exploration of number theory.
6. Mathematicians have long considered number theory to be pure mathematics, but it has important applications to computer science and cryptography studied in the second part of this Chapter

Plan for Part I

Plan for Part I

1. Divisibility and Modular Arithmetic
1.1 Divisibility
1.2 Division
1.3 Congruence Relation

2. Integer Representations and Algorithms
2.1 Representations of Integers
2.2 Base conversions
2.3 Binary Addition and Multiplication

3. Prime Numbers
3.1 The Fundamental Theorem of Arithmetic
3.2 The Sieve of Erastosthenes
3.3 Infinitude of Primes

4. Greatest Common Divisors
4.1 Definition
4.2 Least common multiple
4.3 The Euclidean Algorithm

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Example

Determine whether $3 \mid 7$ holds and whether $3 \mid 12$ holds.

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Example

Determine whether $3 \mid 7$ holds and whether $3 \mid 12$ holds.

Solution: $3 \nmid 7$ but $3 \mid 12$

Properties of divisibility

Theorem

Let a, b, and c be integers, where $a \neq 0$.

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Theorem

Let $a, b,$ *and* c *be integers, where* $a \neq 0$.

1. *If* $a \mid b$ *and* $a \mid c$*, then* $a \mid (b + c)$ *;*
2. *If* $a \mid b$*, then* $a \mid b\ c$ *for all integers* c *;*

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Proof.

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3. *If* $a \mid b$ *and* $b \mid c$, *then* $a \mid c$.

Proof.

1. We prove the first property. Suppose $a \mid b$ and $a \mid c$, then it follows that there are integers s and t with $b = as$ and $c = at$. Hence, $b + c = as + at = a(s + t)$. Hence, $a \mid (b + c)$.

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Let a, b, and c be integers, where $a \neq 0$.

1. *If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;*
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2. Parts (2) & (3) can be proven similarly. Try it as an exercise.

■

Corollary

If a, b, and c are integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ for any integers m and n. (Proof left as exercise)

Plan for Part I

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Theorem ("Division Algorithm")

If a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$*, such that* $a = dq + r$ *(proved in the tutorial.*

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We have: $a \textbf{ div } d = \lfloor \frac{a}{d} \rfloor$.

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Example

1. Quotient and remainder when 101 is divided by 11?

   We have $101 \textbf{ div } 11 = 9$ and $101 \textbf{ mod } 11 = 2$.

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2. Quotient and remainder when 11 is divided by 3?

   We have $11 \textbf{ div } 3 = 3$ and $11 \textbf{ mod } 3 = 2$.

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3. Quotient and remainder when -11 is divided by 3?

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3. Quotient and remainder when -11 is divided by 3?

   We have $-11 \textbf{ div } 3 = -4$ and $-11 \textbf{ mod } 3 = 1$.

Plan for Part I

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1. Determine whether 17 is congruent to 5 modulo 6.

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Example

1. Determine whether 17 is congruent to 5 modulo 6.
   $17 \equiv 5 \mod 6$ because 6 divides $17 - 5 = 12$.

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   $17 \equiv 5 \mod 6$ because 6 divides $17 - 5 = 12$.
2. Determine whether 24 and 14 are congruent modulo 6.

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Example

1. Determine whether 17 is congruent to 5 modulo 6.
   $17 \equiv 5 \mod 6$ because 6 divides $17 - 5 = 12$.
2. Determine whether 24 and 14 are congruent modulo 6.
   $24 \not\equiv 14 \mod 6$ since $24 - 14 = 10$ is not divisible by 6.

More on congruences

Theorem

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

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Proof.

1. If $a \equiv b \mod m$ holds, then (by the definition of congruence) we have: $m \mid a - b$.

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Proof.

1. If $a \equiv b \mod m$ holds, then (by the definition of congruence) we have: $m \mid a - b$.
2. Hence, there is an integer k such that $a - b = km$ holds and equivalently $a = b + km$.

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3. Conversely, if there is an integer k such that $a = b + km$, then we have: $km = a - b$.

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3. Conversely, if there is an integer k such that $a = b + km$, then we have: $km = a - b$.
4. Hence, we have $m \mid a - b$. Thus, $a \equiv b \mod m$ holds.

∎

Relationship between the $\mod m$ and $\mathbf{mod}\ m$ notations

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The relationship between the two notions is stated below:

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Theorem

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \mod m$ *if and only if* $a \mathbf{\ mod\ } m = b \mathbf{\ mod\ } m$ *(See Tutorial.)*

Congruences of sums and products

Theorem

Let a, b, c, d be integers. Let m be a positive integer. If $a \equiv b \mod m$ *and* $c \equiv d \mod m$ *both hold, then we have:* $a + c \equiv b + d \mod m$ *and* $ac \equiv bd \mod m$.

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Proof.

1. Since we have $a \equiv b \mod m$ and $c \equiv d \mod m$, there exist integers s and t with $b = a + sm$ and $d = c + tm$.

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Because $7 \equiv 2 \mod 5$ and $11 \equiv 1 \mod 5$, it follows that:

$$18 = 7 + 11 \equiv 2 + 1 = 3 \mod 5 \text{ and } 77 = 7 \cdot 11 \equiv 2 \cdot 1 = \quad \mod 5.$$

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See the tutorial for a proof.

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(*optional*) Using the terminology of abstract algebra, $\mathbb{Z}_m$ with $+_m$ is a commutative group and $\mathbb{Z}_m$ with $+_m$ and $\cdot_m$ is a commutative ring.

Plan for Part I

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d. *We will prove this using mathematical induction in Chapter 5.*

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a. *Let b be a positive integer greater than 1.*
b. *Then if n is a positive integer, it can be expressed uniquely in the form:*

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

   where k is a non-negative integer, such that $a_0, a_1, \ldots a_k$ are non-negative integers less than b, and $a_k \neq 0$.
c. *The a_j, for $j = 0, \ldots, k$ are called the base-b digits of the representation.*
d. *We will prove this using mathematical induction in Chapter 5.*

2. The representation of n given in the theorem is called the *base b expansion of n* and is denoted by $(a_k a_{k-1} \ldots a_1 a_0)_b$.

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3. We usually omit the subscript 10 for base 10 expansions.

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   Solution: $(1\ 0101\ 1111)_2 =$
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The octal expansion (base 8) uses the digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

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Hexadecimal expansions

The hexadecimal expansion needs 16 digits, but our decimal system provides only 10. So letters are used for the additional symbols. The hexadecimal system uses the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$. The letters A through F represent the decimal numbers 10 through 15.

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   Solution: $1 \cdot 16^2 + 14 \cdot 16^1 + 5 \cdot 16^0 = 256 + 224 + 5 = 485$

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$$q_0 = bq_1 + a_1, \quad 0 \le a_1 < b$$
6. The remainder, a_1, is the second digit from the right in the base b expansion of n.
7. Continuing in this manner (by successively dividing the quotients by b) we obtain the additional base b digits as remainders. The process terminates when a quotient is 0.

continued →

Algorithm: constructing base b expansions

Algorithm 1 base_b_expansion(n, b)

Require: $n, b \in \mathbb{Z}^+,\ b > 1$
Ensure: base b expansion of n: $(a_{k-1} \cdots a_1 a_0)_b$.

$q \leftarrow n$
$k \leftarrow 0$
while $q \neq 0$ **do**
  $a_k \leftarrow q \mod b$
  $q \leftarrow q$ **div** b
  $k \leftarrow k + 1$
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return $(a_{k-1} \cdots a_1 a_0)$

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Find the octal expansion of $(12345)_{10}$

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The remainders are the digits from right to left yielding $(30071)_8$.

Comparison of the hexadecimal, octal, and binary representations

TABLE 1 **Hexadecimal, Octal, and Binary Representation of the Integers 0 through 15.**

Decimal	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hexadecimal	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
Octal	0	1	2	3	4	5	6	7	10	11	12	13	14	15	16	17
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1. Each octal digit corresponds to a block of 3 binary digits.
2. Each hexadecimal digit corresponds to a block of 4 binary digits.
3. So, conversion between binary, octal, and hexadecimal is easy.

Conversion between the binary, octal, and hexadecimal expansions

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1. Find the octal expansion of $(11111010111100)_2$.

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2. Find the hexadecimal expansions of $(11111010111100)_2$.

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Plan for Part I

Binary addition of integers

Algorithms for performing operations with integers using their binary expansions are important as computer chips work with binary numbers. Each digit is called a *bit*.

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Algorithm 2 add (a, b)

Require: $a, b \in \mathbb{Z}^+$, {the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \ldots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \ldots, b_0)_2$, respectively}

Ensure: $(s_n, \ldots, s_1, s_0)$, the addition of a and b. {the binary expansion of the sum is $(s_n, s_{n-1}, \ldots, s_0)_2$ }

$c_{prev} \leftarrow 0$ ▷ represents *carry* from the previous bit addition

for $j \leftarrow 0, n-1$ **do**

  $c \leftarrow \lfloor \frac{(a_j + b_j + c_{prev})}{2} \rfloor$ ▷ quotient (*carry* for the next digit of the sum)

  $s_j \leftarrow a_j + b_j + c_{prev} - 2c$ ▷ remainder (j-th digit of the sum)

  $c_{prev} \leftarrow c$

end for

$s_n \leftarrow c$

return $(s_n, \ldots, s_1, s_0)$

$$a_0 + b_0 = c_0 \cdot 2 + s_0$$
$$a_1 + b_1 + c_0 = c_1 \cdot 2 + s_1$$
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The number of additions of bits used by the algorithm to add two n-bit integers is $\mathcal{O}(n)$.

Binary multiplication of integers

Algorithm for computing the product of two n bit integers.

$$a \cdot b = a \cdot (\quad b_k 2^k \quad + b_{k-1} 2^{k-1} \quad + \dots \quad + b_1 2 \quad + b_0$$

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Algorithm 3 multiply (a, b)

Require: $a, b \in \mathbb{Z}^+$, {the binary expansions of a and b are $(a_{n-1}, a_{n-2}, \dots, a_0)_2$ and $(b_{n-1}, b_{n-2}, \dots, b_0)_2$, respectively}
Ensure: p, the value of ab.

```
for j ← 0, n − 1 do
    if b_j = 1 then
        c_j ← a                                ▷ shifted j places
    else
        c_j ← 0                                ▷ { c_0, c_1, ..., c_{n−1} are the partial products}
    end if
end for
p ← 0
for j ← 0, n − 1 do
    p ← p + c_j
end for
return p {p is the value of ab}
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110	a
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110	ab_0
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```

The number of additions of bits used by the algorithm to multiply two n-bit integers is $\mathcal{O}(n^2)$.

Plan for Part I

Plan for Part I

Primes

Definition

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Example

The integer 7 is prime because its only positive factors are 1 and 7, but 9 is composite because it is divisible by 3.

The fundamental theorem of arithmetic (prime factorization)

Theorem

1. *Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.*

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2. *More formally, for every positive integer a greater than 1, there exists a positive integer n such that there exist prime numbers $p_1, \ldots, p_n$ and positive integers $a_1, \ldots, a_n$ such that:*

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} \quad \text{and} \quad p_1 < p_2 < \cdots < p_n.$$

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Example

1. $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$

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Example

1. $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$
2. $641 = 641$
3. $999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$

The fundamental theorem of arithmetic (prime factorization)

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1. *Every positive integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.*
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Example

1. $100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 \cdot 5^2$
2. $641 = 641$
3. $999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37$
4. $1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$

Plan for Part I

The sieve of Erastosthenes

Erastothenes (276-194 B.C)

The *Sieve of Erastosthenes* can be used to find all primes not exceeding a specified positive integer.

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Example

1. Consider the list of integers between 1 and 100:
   a. Delete all the integers, other than 2, divisible by 2.

The sieve of Erastosthenes

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Example

1. Consider the list of integers between 1 and 100:
   a. Delete all the integers, other than 2, divisible by 2.
   b. Delete all the integers, other than 3, divisible by 3.

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Example

1. Consider the list of integers between 1 and 100:
    a. Delete all the integers, other than 2, divisible by 2.
    b. Delete all the integers, other than 3, divisible by 3.
    c. Next, delete all the integers, other than 5, divisible by 5.

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Example

1. Consider the list of integers between 1 and 100:
   a. Delete all the integers, other than 2, divisible by 2.
   b. Delete all the integers, other than 3, divisible by 3.
   c. Next, delete all the integers, other than 5, divisible by 5.
   d. Next, delete all the integers, other than 7, divisible by 7.

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Example

1. Consider the list of integers between 1 and 100:
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   b. Delete all the integers, other than 3, divisible by 3.
   c. Next, delete all the integers, other than 5, divisible by 5.
   d. Next, delete all the integers, other than 7, divisible by 7.

all remaining numbers between 1 and 100 are prime:
$\{2, 3, 7, 11, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

The sieve of Erastosthenes

Erastothenes (276-

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The *Sieve of Erastosthenes* can be used to find all primes not exceeding a specified positive integer.

Example

1. Consider the list of integers between 1 and 100:
   a. Delete all the integers, other than 2, divisible by 2.
   b. Delete all the integers, other than 3, divisible by 3.
   c. Next, delete all the integers, other than 5, divisible by 5.
   d. Next, delete all the integers, other than 7, divisible by 7.

all remaining numbers between 1 and 100 are prime:
$\{2, 3, 7, 11, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$

Why does this work?

continued →

The sieve of Erastosthenes

TABLE 1 **The Sieve of Eratosthenes.**

Integers divisible by 2 other than 2 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	**2**	**3**	4	**5**	6	**7**	8	9	10
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	43	44	45	46	**47**	48	49	50
51	52	**53**	54	55	56	57	58	**59**	60
61	62	63	64	65	66	**67**	68	69	70
71	72	**73**	74	75	76	77	78	**79**	80
81	82	**83**	84	85	86	87	88	**89**	90
91	92	93	94	95	96	**97**	98	99	100

The sieve of Erastosthenes

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11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
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Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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1. If an integer n is a composite integer, then it must have a prime divisor less than or equal to $\sqrt{n}$.

The sieve of Erastosthenes

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1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. If an integer n is a composite integer, then it must have a prime divisor less than or equal to $\sqrt{n}$.
2. To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

The sieve of Erastosthenes

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Integers divisible by 2 other than 2 receive an underline.

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11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
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51	52	53	54	55	56	57	58	59	60
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 3 other than 3 receive an underline.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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21	22	23	24	25	26	27	28	29	30
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41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
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71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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1. If an integer n is a composite integer, then it must have a prime divisor less than or equal to $\sqrt{n}$.
2. To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
3. For n = 100, $\sqrt{n} = 10$, thus any composite integer ≤ 100 <u>must have</u> prime factors less than 10, that is 2,3,5,7. The remaining integers ≤ 100 are prime.

The sieve of Erastosthenes

TABLE 1 The Sieve of Eratosthenes.

Integers divisible by 2 other than 2 receive an underline.

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Integers divisible by 3 other than 3 receive an underline.

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21	22	23	24	25	26	27	28	29	30
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91	92	93	94	95	96	97	98	99	100

Integers divisible by 5 other than 5 receive an underline.

1	2	3	4	5	6	7	8	9	10
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21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
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51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
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81	82	83	84	85	86	87	88	89	90
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Integers divisible by 7 other than 7 receive an underline; integers in color are prime.

1	2	3	4	5	6	7	8	9	10
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21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
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1. If an integer n is a composite integer, then it must have a prime divisor less than or equal to $\sqrt{n}$.
2. To see this, note that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
3. For n $= 100$, $\sqrt{n} = 10$, thus any composite integer ≤ 100 must have prime factors less than 10, that is 2,3,5,7. The remaining integers ≤ 100 are prime.
4. *Trial division*, a very inefficient method of determining if a number n is prime, is to try every integer $i \leq \sqrt{n}$ and see if n is divisible by i.

Plan for Part I

Infinitude of primes

Euclid (325 - 265 B.C)

Theorem

There are infinitely many primes.

Infinitude of primes

Euclid (325 - 265 B.C)

Theorem

There are infinitely many primes.

PROOF.

1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.

Infinitude of primes

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There are infinitely many primes.

PROOF.

1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.
2. Let $q = p_1 p_2 \cdots p_n + 1$

Infinitude of primes

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PROOF.

1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.
2. Let $q = p_1 p_2 \cdots p_n + 1$
3. Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.

Infinitude of primes

Euclid (325 - 265 B.C)

Theorem

There are infinitely many primes.

Proof.

1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.
2. Let $q = p_1 p_2 \cdots p_n + 1$
3. Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
   a. If a prime p_j divides q, and since $p_j \mid p_1 p_2 \cdots p_n$ holds as well, then p_j divides $q - p_1 p_2 \cdots p_n = 1$.

Infinitude of primes

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There are infinitely many primes.

PROOF.

1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.
2. Let $q = p_1 p_2 \cdots p_n + 1$
3. Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
   - a. If a prime p_j divides q, and since $p_j \mid p_1 p_2 \cdots p_n$ holds as well, then p_j divides $q - p_1 p_2 \cdots p_n = 1$.
   - b. Thus, if a prime p_j divides q, then $p_j = 1$, which is a contradiction with $p_j > 1$.

Infinitude of primes

Euclid (325 - 265 B.C)

Theorem

There are infinitely many primes.

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1. Assume finitely many primes: $p_1, p_2, \ldots, p_n$.
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3. Either q is prime or by the fundamental theorem of arithmetic it is a product of primes.
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   b. Thus, if a prime p_j divides q, then $p_j = 1$, which is a contradiction with $p_j > 1$.
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This proof was given by Euclid in *The Elements* .

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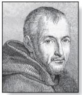
6. Fortunately, we can generate large integers which are almost certainly primes.

Mersenne primes



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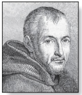

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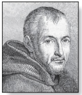

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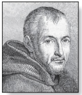

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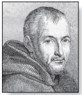

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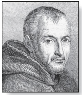

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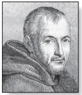

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http://www.mersenne.org/

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3. **The Twin Prime Conjecture**: there are infinitely many pairs of twin primes. Twin primes are pairs of primes that differ by 2. Examples are 3 and 5, 5 and 7, 11 and 13, etc. The current world's record for twin primes (as of mid 2011) consists of numbers $65,516,468,355 \cdot 23^{33,333} \pm 1$, which have 100,355 decimal digits.

Plan for Part I

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2. Determine whether the integers 10, 19, and 24 are pairwise relatively prime.
   Solution: No, since $\gcd(10, 24) = 2$.

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$$\gcd(120, 500) = 2^{min(3,2)} \cdot 3^{min(1,0)} \cdot 5^{min(1,3)} = 2^2 \cdot 3^0 \cdot 5^1 = 20$$

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Remark: finding the GCD of two positive integers using their prime factorizations is not efficient because there is no efficient algorithm for finding the prime factorization of a positive integer.

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$\mathrm{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{max(3,4)} 3^{max(5,3)} 7^{max(2,0)} = 2^4 3^5 7^2$

Theorem

Let a and b be positive integers. Then, we have:

$$a \cdot b = \gcd(a, b) \cdot \mathrm{lcm}(a, b)$$

Plan for Part I

The Euclidean Algorithm

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1. Find $\gcd(287, 91)$:

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   a. $287 = 91 \cdot 3 + 14$

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1. Find $\gcd(287, 91)$:
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$$\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = \gcd(7, 0) = 7$$

continued →

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The Euclidean algorithm expressed in pseudo-code is:

Algorithm 4 $\mathrm{gcd}(a, b)$

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```
x ← a
```

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Note: the time complexity of the algorithm is $\mathcal{O}(\log^2 a)$, where $a > b$.

Correctness of the Euclidean Algorithm

Lemma

Let $r = a \textbf{ mod } b$ *, where* $a \geq b > r$ *are integers. Then, we have:*

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4. Therefore, $\gcd(a, b) = \gcd(b, r)$.

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1. Suppose that a and b are positive integers with $a \geq b$. Let $r_0 = a$ and $r_1 = b$. Successive applications of the division algorithm yields:

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2. Eventually, a remainder of zero occurs in the sequence of terms: $a = r_0 \geq r_1 > r_2 > \cdots \geq 0$. The sequence can not contain more than $(a+1)$ terms.
3. Then, the Lemma implies: $\gcd(a, b) = \gcd(r_0, r_1) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n$.
4. Hence the GCD is the last nonzero remainder in the sequence of divisions . ■

GCD(s) as linear combinations

Étienne Bézout

(1730 - 1783)

Theorem (Bézout's Theorem)

If a and b are positive integers, then there exist integers s and t such that

$$\gcd(a, b) = sa + tb.$$

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1. If a and b are positive integers, then integers s and t such that $\gcd(a, b) = sa + tb$ are called *Bézout coefficients* of a and b.

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2. The equation $\gcd(a, b) = sa + tb$ is called *Bézout's identity.*
3. The expression $sa + tb$ is also called a *linear combination* of a and b with coefficients of s and t.

Example

$\gcd(6, 14) = 2 = (-2) \cdot 6 + 1 \cdot 14$

Finding GCD(s) as linear combinations

Example

Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

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Example

Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show $\gcd(252, 198) = 18$

a. $252 = 1 \cdot 198 + 54$

Finding GCD(s) as linear combinations

Example

Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show $\gcd(252, 198) = 18$

a) $252 = 1 \cdot 198 + 54$
b) $198 = 3 \cdot 54 + 36$

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Express $\gcd(252, 198) = 18$ as a linear combination of 252 and 198.

Solution: First use the Euclidean algorithm to show $\gcd(252, 198) = 18$

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b. $198 = 3 \cdot 54 + 36$
c. $54 = 1 \cdot 36 + 18$

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b. $198 = 3 \cdot 54 + 36$
c. $54 = 1 \cdot 36 + 18$
d. $36 = 2 \cdot 18$

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- a $252 = 1 \cdot 198 + 54$
- b $198 = 3 \cdot 54 + 36$
- c $54 = 1 \cdot 36 + 18$
- d $36 = 2 \cdot 18$

1. Working backwards , from c and b above

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$$18 = 54 - 1 \cdot 36$$

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   $18 = 54 - 1 \cdot 36$

   $36 = 198 - 3 \cdot 54$

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   $18 = 54 - 1 \cdot 36$

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3. Substituting $54 = 252 - 1 \cdot 198$ (from (a) above) yields:

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3. Substituting $54 = 252 - 1 \cdot 198$ (from a above) yields:

   $18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198$

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3. Substituting $54 = 252 - 1 \cdot 198$ (from a above) yields:

$$18 = 4 \cdot (252 - 1 \cdot 198) - 1 \cdot 198 = 4 \cdot 252 - 5 \cdot 198$$

This method illustrated above is a two pass method. It first uses the Euclidean algorithm to find the GCD and then works backwards to express the GCD as a linear combination of the original two integers. There is a one pass method, called the *extended Euclidean algorithm.*

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If a, b, c are positive integers such that a and b are relatively prime (that is, $\gcd(a, b) = 1$) and $a \mid bc$, then we have $a \mid c$.

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A generalization of the above lemma is important in practice:

Lemma

If p is prime and $p \mid a_1 a_2 \ldots a_n$ where a_i are integers then $p \mid a_i$ for some i.

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2. With the previous lemma and since $\gcd(c, m) = 1$ holds, it follows that $m \mid a - b$..
3. Hence, $a \equiv b \mod m$.

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