What is an Equation?

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***Abstract*—The equation concept is one of the most fundamental in mathematics, yet the word "equation" means different things to different people. It is used not only with various precise technical meanings, but also as a metaphor for complex situations. We review some of the history of the equation and its present meaning and use in a variety of settings. Although some languages make distinctions among ideas such equations that are always satisfied versus those that may not be, or between equations with variables and relations without, we observe that it may not be possible to decide into which of these cases a particular equality falls.**

Keywords—equation; history of mathematics; mathematical terminology, mathematical translation

I. EQUATIONS EVERYWHERE

There are few concepts in modern mathematics that are more fundamental or more used than that of the equation. The term is so basic that most users of mathematics, and indeed most mathematicians, seldom think about what it means. We often discuss equations without explicitly establishing the domain of discourse and we come to conclusions without ever agreeing what we are talking about.

This mathematical word now transcends its technical meaning and has taken hold in everyday language as a metaphor for complex relationships or as an appeal for an act of solution. Two years ago, U.S. President Barack Obama launched the slogan *Change the Equation*, in order to stimulate innovative programs in mathematics and in science. In mass-media, in advertising, the word *equation* is used as a symbol for some problematic situations, asking for the right answer.

Nobody could exclude Chancellor Merkel from the political equation of the next years; the astrological equation of personality; the equation of life and death; Federer's equation; the equation of a quiet life; the equation of success: work + risk + tenacity; money is included in the equation of happiness; Annan and Syria's insoluble equation; human equation; the project Equation has been launched at 29 November 2011; *Equation* – an integrated, independent Belgian communication agency; title of a film: *The love in equation; Afghanistan's role in the equation of big powers.*

These are only a few examples turned out by a cursory web search. They show something about the public perception of mathematics. Just because it is difficult for most people to understand mathematics, these people infer that it deserves to be admired and considered as a universal pattern. But at the same time and for the same reason, many people feel intimidated by mathematics and they are ready to expect from it much more than it can really done. On the other hand, the lack of understanding generates sometimes the opposite effect, the suspicion that mathematics manipulates people; a doubt emerges about the real utility of mathematics. This doubt is visible in the attitude of many governments, which claim from mathematics immediate practical applications, in contrast with its very nature: mathematics is a long distance enterprise; in most cases, its impact is not visible next day. But we can see today's impact of mathematics done some decades, centuries and millennia ago. The word *equation*, by its multiple aspects and by its huge metaphorical capacity, concentrates and articulates to a large extent several controversial situations, symptomatic for the ambiguous status of mathematics.

II. EQUATIONS AS A TRAP

Until two years ago, both of us were convinced that *equation* in English had the same meaning as *équation* in French, *Gleichung* in German and *ecuaţie* in Romanian. This belief stopped in September 2010, when Marcus was shocked by the statement at page *xiv* of the book *Where Mathematics Comes From* [1], claiming that Euler's famous relation between e, π, and i is an equation, despite the fact that it includes no variable. How is it possible to have such a mistake in a famous book, opening new ways in connecting cognitive sciences and mathematics? This was Marcus' state of mind at that moment.

As it happened, this observation occurred in Timişoara during the 2010 International Symposium on Symbolic and Numeric Algorithms for Scientific Computing. When Marcus expressed this wonder to Watt, his reply came as another shock: "I am sorry, in English any equality is an equation." Accross from us was a professor from Japan, who confirmed what Watt said.

Back in our home cities, we both began to pursue the question further. For *Larousse* and *Encyclopaedia Universalis* only equalities including one or several variables can be equations, while in English dictionaries any equality is an equation, as Marcus confirmed from Professor Sergiu Rudeanu. Watt took this relay further, interacting with people of different countries, bringing a more detailed picture of the situation. But the great surprise came when we approached the Latin

Equation, (in *Algebra*) a mutual comparing of two equal Quantities, or Things of different Names, or Kinds; as when the value of 3 Shillings is compared to 36 Pence.

In *Aſtronomy*, Equation is taken for the proportioning, or regulating of Time, or the difference between the Time mark'd out by the Sun's apparent Motion, and the Time that is meaſur'd by its real, or middle Motion, according to which Watches and Clocks are to be adjuſted.

Equation or Optical Proſtaphæreſis, (in the *Ptolemaick* Syſtem) is the Angle made by two Lines drawn from the Center of the *Epicycle*, to the Centers of the World, and of the *Eccentrick*.

Equation or Phyſical Proſtaphæreſis, is the difference between the Motions of the Center of the *Epicycle*, in the *Equant*, and in the *Eccentrick*.

Equation or Total Proſtaphæreſis, is the difference between the Planet's mean and true Motion, or the Angle made by the Lines of the true and mean Motion of the Center.

Fig. 1. Entries in John Kersey's dictionary of 1708.

etymology of the word *equation* and we learned from Professor Mihai Dinu (Faculty of Letters, University of Bucharest) that in Quicerat and Devaluy's *Latin-French Dictionary* [2] it is clearly stated that *Aequatio, -onis = égalité*. In other words, in Latin, the word *equation* has exactly the meaning of its Latin source: *equality*. On the other hand, against expectations, Romanic languages, that is languages having Latin as their root, did not remain faithful to the original meaning of the respective word. As examples, in Quicherat-Devaluy's Dictionary, we find: *honorum = égalité de crédit, partage égal des biens* (Cicero) and also, from the field of jurisprudence, *égalité de droit* (Titus Livius).

A clear confirmation of the gap between two different ways to understand the word *equation* is given now by Wikipedia, where in the English version we find *An equation is a mathematical statement that asserts the equality of two expressions*, while in the French version we find *Une equation est en mathématique une égalité contenant une ou plusieurs variables*. It immediately follows to ask what does it mean *to solve* an equation?

Going now back to the huge metaphorical use of *equation*, one can observe that the examples we gave at the beginning are of three types. Some of them follow the English meaning (for example: *the equation of success*) and they equate *equation* with *formula* or with *prescription*; other examples follow the French meaning (the clearest example in this respect is that related to *Syria's insoluble equation*). But there are also situations where ambiguity prevails and we can accept both meaning: *the political equation of the next years* is clearly of this type. As a matter of fact, many other situations occur. For instance, in the syntagm *chemical equation* no equality appears, we have only a symbolic expression of a chemical reaction, i.e., of a transformation expressed by an arrow.

III. The Word "Equation" in English

While the word *equation* derives from the Latin *aequationem*, its meaning in English has evolved from its original introduction into the language in the late 14th century.

According to the Oxford English Dictionary [3], the early uses of the term *equation* were in the area of astrology and astronomy. These uses referred to making equal partitions of various celestial quantities, such as the method of dividing the sphere equally into 'houses'. The word *equation* occurs already in 1391 in Chaucer's *A Treatise on the Astrolabe* [4], the astrolabe being a device for measuring inclinations in celestial observations,

> *With the smale point of the foreside label, shalltow kalcule thyne equaciouns in the bordure of thin Astrolabie.*

and later in his Canterbury tales[5]:

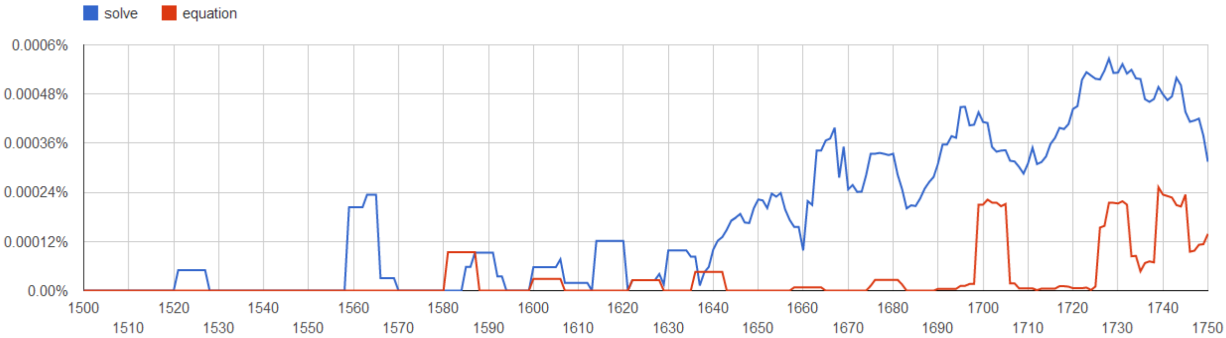
> *And hise proporcionels conuenientz*
> *For hise equacions in euery thyng.*

The mathematical use of the word *equation*, meaning the action of stating the equality of two quantities, appears to date from the late 16th century. For example, the Oxford English Dictionary observes the use in Dee's preface to Henry Billingsley's translation of Euclid's *Elements* [6]

> *Which thing, is well to be perceiued in that*
> *great Arithmeticall Arte of Æquation:*
> *commonly called the Rule of Coss. or Algebra.*

The use of the word in a general sense for the action of making equal or balancing is seen from the mid-17th century, e.g. [7]

> *The very Redundance it self of Mankind seeming by a natural consecution to yield and subminister this Remedy, for its Reduction and Equation.*



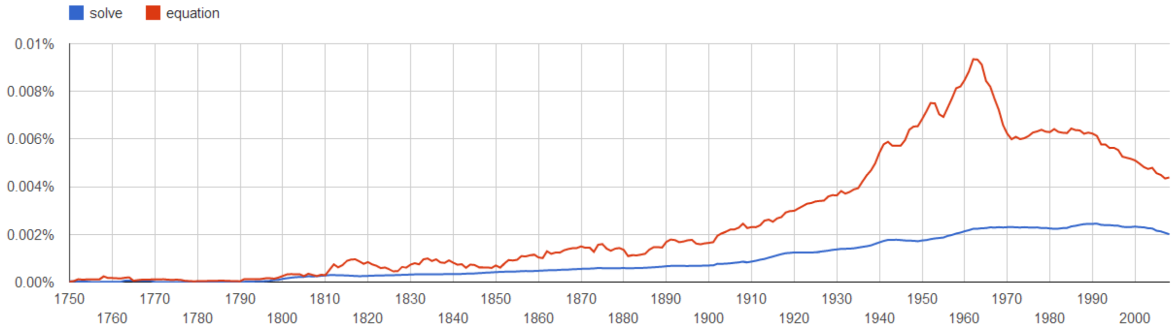


Fig. 2. Relative frequencies of "equation" and "solve" in English. The top graph shows from 1500 to 1750 [Google UK English Database], and the bottom graph from 1750 to 2008 [Google English 1 Million Database].

In this general sense, the term *equation* was used chiefly in phrases of the form "equation of …", e.g. "equation of international demand". Although this more general usage was spreading, the word continued to be used significantly in its astronomical senses. John Kersey's dictionary [8], published at the beginning of the 18th century, gives one meaning related to what he categorized as algebra, but then goes on to give several distinct astronomical meanings. This is shown in Figure 1. In the time since its introduction, the use of the word *equation* has increased in frequency relative to other words in English. Two illustrative charts, based on Google's database of English language works, are shown in Figure 2.

As the field of algebra has developed, the concept of variables or unknowns in equations has been introduced and has evolved. This is naturally tied to the notion of solving for these unknowns. Figure 2 also shows the rise of the relative use of the word "solve".

While the astronomical uses of the word *equation* have declined, the technical mathematical use has risen. But beyond this, we see the word used not only in its original sense of equating quantities, but also in a figurative sense when dealing with unknowns or with something to solve. For example, The American HeritageDictionary [9] has as one definition

> *a situation, esp one regarded as having a number of conflicting elements: what you want doesn't come into the equation*

The mathematical sense of the word *equation* has become inextricably connected to the use of a particular symbol. For example, the Oxford Dictionary of English now gives as its first definition [10].

The use of the equality symbol has an interesting history, to which we turn next.

IV. THE EQUATION IN MATHEMATICAL NOTATION

There were several early symbols used to indicate equality, a very nice survey of which is given by Cajori [11] and whose principal points we summarize here. Another useful reference is that of Babbage [12], the existence of which underscores the link between mathematical notation and computation.

Early documents have various symbols or notations to say that a computation gives a certain value. This is seen as far back as the *Rhind papyrus*, a mathematical document copied by the scribe Ahmes from an older document between 1700 and 1550 BCE. Figure 3 shows the symbol  used as "it gives".

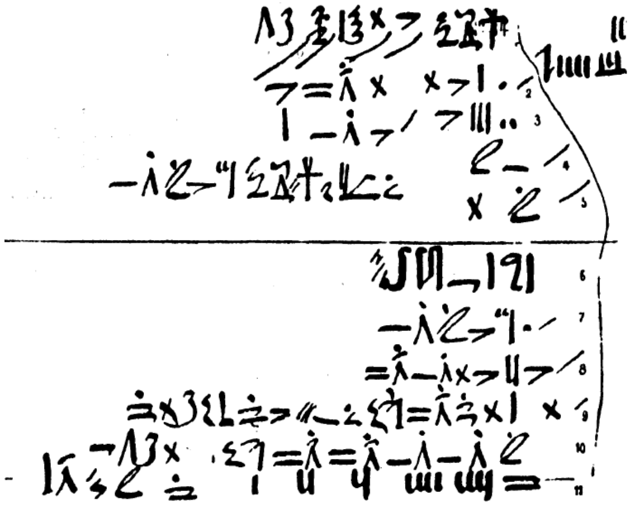


Fig. 3. An algebraic equation and its solution in the Rhind papyrus (from [11]). The “it gives” symbol is seen second from the right in the top line.

In the original manuscripts of Diophantus, equality seems to have been indicated by the symbol ι^{σ}, although subsequent transcriptions have not always copied the notation faithfully.

Another example appears in the Bakshālī manuscript, found in what is now North West India in 1881. This document is an incomplete copy of an 8th to 10th century manuscript written in the Śāradā script in the Gatha dialect, related to Sanskrit and Prakrit. This document indicates equality of a computational result with *pha,* an abbreviation for *phala,* which is Sanskrit for “fruit”.

Fifteenth century Arabic mathematics produced Al-Qalasâdî’s *Raising the Veil of the Science of Gubar*. In this, he used the symbol ⅃ as the equality sign. Some contemporary European authors, such as Regiomontanus and Pacioli, used dashes to indicate equality. In the century that followed, it was usual in printed books to express equality using words such as *aequales, aequantur, esgale, faciunt, ghelijck, gleich* or in abbreviated form, e.g. *aeq.*

We trace the use of the modern equality symbol “=” to Robert Recorde in 1557. Recorde was a Welsh physician and mathematician born almost exactly five centuries ago. He studied first at Oxford and then at Cambridge. He taught mathematics at Oxford, served as a royal physician, was controller of the Royal mint and held a number of other positions.

Recorde’s book, *The Whetstone of Witte* [13], introduced the modern equality symbol, and was the first in English to use the modern plus and minus signs. This work is now seen as having brought algebra to England. It covered an array of arithmetic topics, and was correspondingly subtitled to position it as a sequel to Diophantus’ *Arithmetica*. Figure 4

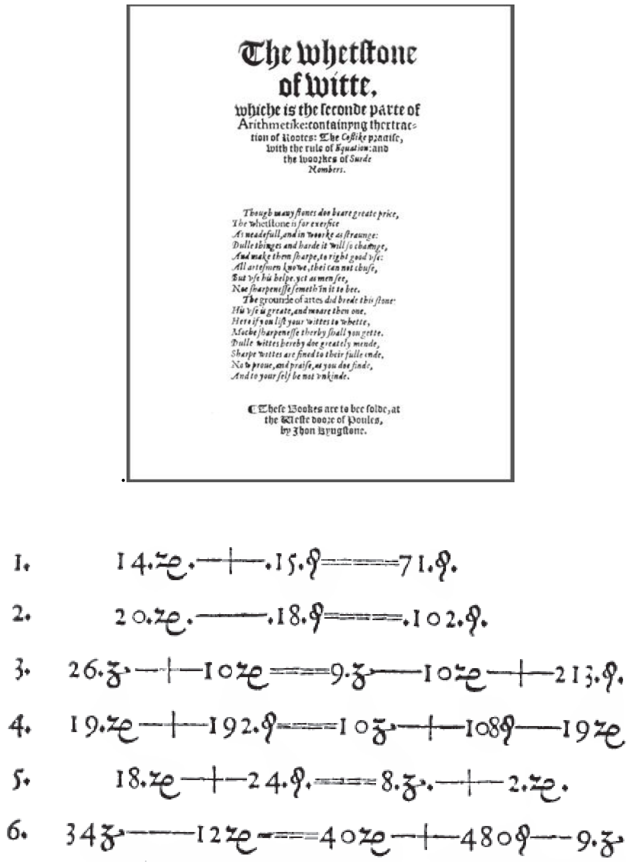


Fig. 4. Equations in the Whetstone of Witte. The first equation would be $14x + 15 = 71$ using modern notation for variables.

shows the title page and examples of equations from its section entitled *The rule of equation, commonly called Algebers Rule*. Recorde introduced the modern equality symbol as follows

> *And to auoide the tediouse repetition of these woordes :is equalle to: I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus: =, bicause noe .2. thynges, can be moare equalle.*

Following its introduction, the modern equality symbol did not appear again in print as such for several decades, although it was used in private manuscripts and letters. In the meantime, the symbol was confusingly used for different purposes in Europe, including François Viète’s use for arithmetical difference and René Descartes’ use for “±”. Additionally, Johann Caramuel used the symbol as the decimal separator, i.e. his 3=14 would be our 3.14. Others used “=” in geometry to indicate parallel lines. After a period of confusion, in which various meanings were ascribed to “=” and a variety of notations were used for equality, Recorde’s “=” came into common use in England. It was used by Thomas Harriot, John Wallis, Isaac Barrow and Isaac Newton and spread to Europe. Examples of the same equation in different notations leading up to modern notation are shown in Figure 5.

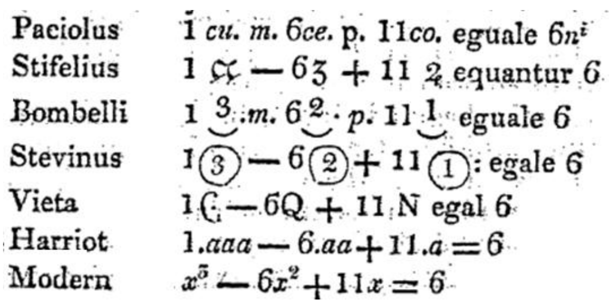


Fig. 5. Different forms of the same equation over time (from [12])

V. TYPES OF EQUATIONS AND TERMINOLOGY IN VARIOUS LANGUAGES

Having seen the history of the word *equation* and the use of the equality symbol "=", we now return to the different meanings of the word and its cognates in different languages. Recall that the discussion started around the point of whether a mathematical equality without variables should be called an equation. Here we see a difference in accepted practice in different languages. Now we see this is not surprising, as the word *equation* has become established before the "=" notation and well before the modern concept of variables.

We have conducted a small survey of accepted use of "equation" and related words in different languages. We may have a number of cases:

Type 1. An equality with variables over some domain (e.g. $\mathbb{N}, \mathbb{R}$ or $\mathbb{C}$) where some, but not all, value assignments for the variables make the equality true, e.g. $x^4 - 1 = 0$ or $x^2 + y^2 = 25$, with x and y real-valued.

Type 2.An equality with variables over some domain (e.g. $\mathbb{N}, \mathbb{R}$ or $\mathbb{C}$) where all value assignments for the variables make the equality true, e.g. $x + y = y + x$, with x and y real-valued.

Type 3. An equality with variables over some domain (e.g. $\mathbb{N}, \mathbb{R}$ or $\mathbb{C}$) where no value assignment for the variables makes the equality true, e.g. $x^2 + y^2 + 1 = 0$, with x and y real-valued.

Type 4. An equality statement with no variables that is true, e.g. 8 + 2 = 10 or $e^{i\pi} + 1 = 0$.

Type 5. An equality statement with no variables that is false, e.g. 8 + 2 = 9 or $0\,x + 2 = 9$, with x a real-valued variable.

We are assuming for this discussion that the domains and co-domains of the functions are all the same.

Some results are summarized in Figure 6. We see that there are indeed a variety of different patterns of common use. Additional languages, including Armenian and Persian, showed usage similar either to English or French. Some languages allow one word for all, others require different words for each. Some have alternatives or modifiers to cover different subsets of the cases. In most cases there are general terms to cover classes of expressions broader than equalities, such as "expression" or "formula".

As we noted earlier, we find it interesting that the word *equation* in English comes from Latin and both English and Latin allow it to be used broadly, but French and Romanian, which are more closely related to Latin, do not.

VI. THE ILLUSION OF EQUATION TYPES

The preceding discussion shows how different forms of equality relations can be considered differently in different languages. We now observe that making these distinctions is more difficult than it at first appears.

Daniel Richardson showed [14], more than 50 years ago, that under certain relatively easy conditions the problem of determining the equivalence of an expression to zero is generally unsolvable. This can be used to show that no algorithm can reliably recognize the distinctions made among these various forms of equation.

More specifically Richardson shows the following: Let E be a set of expressions for real, single-valued, partially-defined functions of a real variable and let E^* be the set of functions represented by expressions in E. Write $A(x)$ for the function in E^* denoted by the expression A in E. Richardson requires that E^* contains the identity function, the rational numbers (as constant functions) and that it be closed under addition, subtraction, multiplication and functional composition. It is also presumed that given expressions A and B there is an effective procedure to find an expression to represent the sum, difference, product and functional composition of the corresponding functions. Richardson shows that if E^* contains $\log 2, \pi, e^x$ and $\sin x$, and a function $\mu(x) = |x|$ for $x \neq 0$, then it is not possible to decide, given A in E whether $A(x)$ is defined and everywhere equal to 0.

Subsequent papers addressed expression simplification in this context [15], the undecidability of the existence of zeros of real elementary functions [16] and classes of expressions where zero equivalence is decidable [17].

	$x^4 - 1 = 0$	$x + y = y + x$	$8 + 2 = 10$	$E = mc^2$
Arabic	معادلة	مطابقة	علاقة	
Chinese	方程	恒等式	等式	
English	equation	equation identity	equation identity	equation law, formula
French	équation	identité	relation	
German	Gleichung	(identische) Gleichung Identitäte	(identische) Gleichung Identitäte	
Greek	εξίσωση ισοτητα	ταυτότητα ισοτητα	σχέση ισοτητα	νόμος
Hebrew	משוואה	זהות		
Latin	aequatio	aequatio	aequatio	
Romanian	ecuaţie	identitate	relaţie, egalitate	relaţie
Russian	уравнение	тождество	равенство соотношение	

Fig. 6. Examples of equations in different languages

Richardson's result on the undecidability of zero equivalence implies that we cannot always know whether a sub-expression involving a variable is identically zero. It is therefore easy to construct cases that cannot be decided between types 1,2 and 3 in the previous section's classification. For example, consider the family of equalities given by the following, with x real-valued:

$$E(x) \times (x^2 - 1) + x + 1 = 1 + x.$$

If $E(x)$ is equivalent to zero, then we have an equality of type 2 – what we would call in French an *identité*. On the other hand, if $E(x)$ is somewhere non-zero, then we have an equality of type 1 or 3 – what we would call in French an *équation*. So there generally is no procedure to distinguish between types of equalities with variables.

When it comes to matters that depend on whether an equality has variables, things are similar. Does the equality

$$0 \times y + 1 = 3 - 2$$

contain a variable? What if the 0 were replaced by $E(x)$ that is equivalent to zero? The best we can say is that an equation contains a variable syntactically. We cannot tell whether an equation relates two constant functions or two non-constant functions. Finally, in the case where the two sides are constant functions, it is not generally decidable whether they are equal. Thus it is not always possible to distinguish types 4 and 5.

VII. Turning to Bourbaki

Ambiguities are not limited to the strictly mathematical use of the term *equation* – figurative use finds these as well. For example, one sees an interesting use of *equation* in the article, "Bourbaki, l'équation collective" [18]. As this is in French, we may ask where the variables are. After a careful reading, one may notice that the word *equation* appearing in the title never appears in the text. It is an ambiguity whether its metaphorical use should be considered as equivalent to a formula or to something to be solved. There are things in the article which favour the first choice, for instance the description of the way the members of Bourbaki used to work, according to some pre-established rules (one of them, to leave the group when you are fifty). But there are other aspects, favouring the understanding of *equation* as something to be solved:

- The difficulty to decide whether Bourbaki is still alive or, if not, what is the date of its death;
- There are also debates about the real members of the group, to be distinguished from the friends or temporary associates;
- Are there non-French members of the group? For instance, Saunders MacLane?
- What is the real heritage of Bourbaki? What was positive and what negative in its accomplishments?

So, in this case, like many others, both meanings of *equation* should be considered.

VIII. CONCLUSION

Our considerations are only the beginning of a direction that could take in consideration many other idioms and should try to explain when and why happened the divorce between French and Latin in the way they understand the same word; when and why other languages selected one or the other meaning; what other words followed a similar surprising evolution. What is the real size of discrepancy among English and Romance mathematical terminology? We have an intellectual trans-disciplinary exercise, bridging mathematics, linguistics anthropology and history.

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