

Counting

Chapter 6

With Question/Answer Animations

Chapter Summary

- The Basics of Counting
- The Pigeonhole Principle
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

The Basics of Counting

Section 6.1

Section Summary

- The Product Rule
- The Sum Rule
- The Subtraction Rule (Inclusion-Exclusion)

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into **a sequence of two (or more) tasks**. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

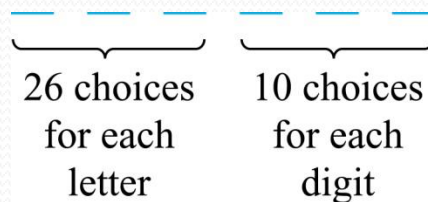
Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$ different possible license plates.



Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: Suppose the elements in the domain are a_1, a_2, \dots, a_m . There are n ways to choose the value of a_1 and $n-1$ ways to choose a_2 , etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Telephone Numbering Plan

Example: The *North American numbering plan (NANP)* specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was $NYX-NNX-XXXX$.
- In the new plan, the format is $NXX-NXX-XXXX$.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX .
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX .
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX .
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10,000$ station codes with the format $XXXX$.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10,000 = 1,024,000,000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10,000 = 6,400,000,000$.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$.
(In Section 5.1, mathematical induction was used to prove this same result.)

Solution: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length $|S|$. When the i -th element is in the subset, the bit string has a 1 in the i -th position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

- If A_1, A_2, \dots, A_m are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

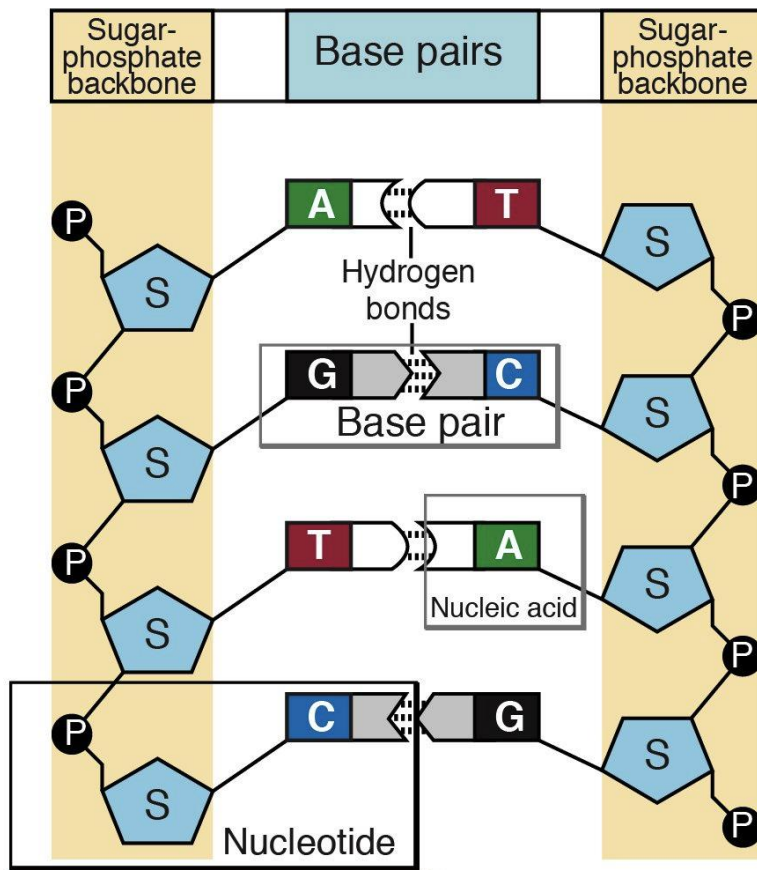
Indeed:

- The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \dots \times A_m$ is done by choosing an element in A_1 , an element in A_2 , ..., and an element in A_m .
- By the **product rule**, it follows that:

$$|A_1 \times A_2 \times \dots \times A_m| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_m|$$

DNA and Genomes

Deoxyribonucleic Acid (DNA)



- A** Adenine
- T** Thymine
- C** Cytosine
- G** Guanine

A gene (DNA) can be abstractly represented as a **string** with elements from the alphabet

$$\Sigma = \{A, T, C, G\}$$

e.g. **AGTCTCCATGAAGCACGTTTAC...**

DNA and Genomes

- A *gene* is a segment of a DNA molecule that encodes a particular protein. The entirety of genetic information of an organism is called its *genome*.
- The DNA of bacteria has between 10^5 and 10^7 *nucleotides* (**one of the four bases**). Mammals have between 10^8 and 10^{10} nucleotides. So, **by the product rule** there are at least 4^{10^5} different sequences of bases in the DNA of bacteria and 4^{10^8} different sequences of bases in the DNA of mammals.
- The human genome includes approximately 23,000 genes, each with 1,000 or more nucleotides.
- Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The Sum Rule in terms of sets.

- The sum rule can be phrased in terms of sets.

$|A \cup B| = |A| + |B|$ as long as A and B are disjoint sets.

- Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$

when $A_i \cap A_j = \emptyset$ for all i, j .

- The case where the sets have elements in common will be discussed when we consider the subtraction rule

Combining the Sum and Product Rule

Example: Suppose statement labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

Solution: Use the sum and product rules.

$$26 + 26 \cdot 10 = 286$$

Counting Passwords

- Combining the sum and product rule allows us to solve more complex problems.
Example: Each user on a computer system has a password, which is **six to eight characters long**, where each character is an **uppercase letter or a digit**. Each password must contain **at least one digit**. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6 , P_7 , and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6 , P_7 , and P_8 , we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:

$$\begin{aligned}P_6 &= 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560. \\P_7 &= 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920. \\P_8 &= 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.\end{aligned}$$

Consequently, $P = P_6 + P_7 + P_8 = 2,684,483,063,360$.

Internet Addresses

- Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31				
Class A	0	netid					hostid						
Class B	1	0	netid					hostid					
Class C	1	1	0	netid					hostid				
Class D	1	1	1	0	Multicast Address								
Class E	1	1	1	1	0	Address							

- **Class A Addresses:** used for the largest networks, a 0, followed by a 7-bit netid and a 24-bit hostid.
- **Class B Addresses:** used for the medium-sized networks, a 10, followed by a 14-bit netid and a 16-bit hostid.
- **Class C Addresses:** used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.
- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A , x_B , and x_C denote the number of addresses for the respective classes.

- To find, x_A : $2^7 - 1 = 127$ netids. $2^{24} - 2 = 16,777,214$ hostids.
 $x_A = 127 \cdot 16,777,214 = 2,130,706,178$.
- To find, x_B : $2^{14} = 16,384$ netids. $2^{16} - 2 = 16,534$ hostids.
 $x_B = 16,384 \cdot 16,534 = 1,073,709,056$.
- To find, x_C : $2^{21} = 2,097,152$ netids. $2^8 - 2 = 254$ hostids.
 $x_C = 2,097,152 \cdot 254 = 532,676,608$.
- Hence, the total number of available IPv4 addresses is

$$\begin{aligned}x &= x_A + x_B + x_C \\ &= 2,130,706,178 + 1,073,709,056 + 532,676,608 \\ &= 3,737,091,842.\end{aligned}$$

Not Enough Today !!

The newer IPv6 protocol solves the problem of too few addresses.

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done **either in one of n_1 ways or in one of n_2 ways**, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

- Also known as, the *principle of inclusion-exclusion*:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$

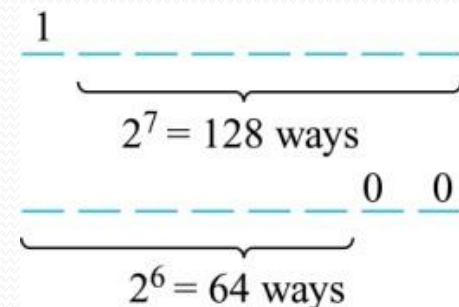
$$\frac{1}{\underbrace{\hspace{10em}}_{2^7 = 128 \text{ ways}}}$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$

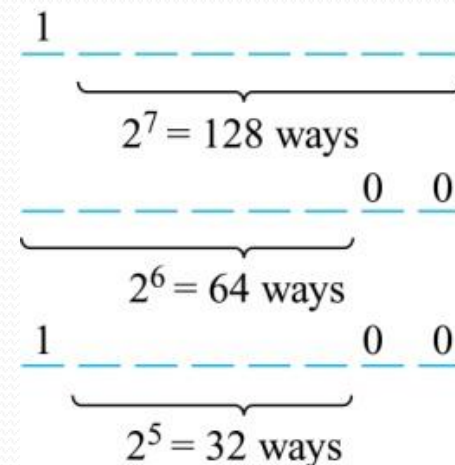


Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.

- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$



Hence, the number is $128 + 64 - 32 = 160$.

The Pigeonhole Principle

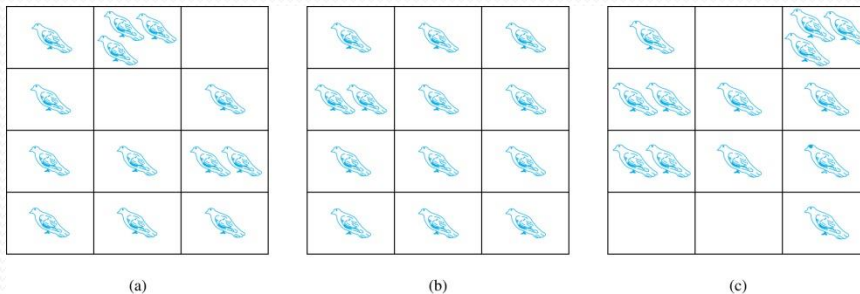
Section 6.2

Section Summary

- The Pigeonhole Principle
- The Generalized Pigeonhole Principle

The Pigeonhole Principle

- If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



Pigeonhole Principle: If $k + 1$ objects (for $k > 0$) are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contraposition. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k . This contradicts the statement that we have $k + 1$ objects.



The Pigeonhole Principle

Corollary 1: A function f from a set with $k + 1$ elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the *codomain* of f .
- Put in these boxes all of the elements x from the domain such that $f(x) = y$.
- Because there are $k + 1$ elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one.



Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example: Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer. Consider the $n + 1$ integers $1, 11, 111, \dots, 11\dots1$ (where the last has $n + 1$ bits). There are n possible remainders when an integer is divided by n . By the pigeonhole principle, when each of the $n + 1$ integers is divided by n , at least **two must have the same remainder**. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then the total number of objects is at most

$$k \left(\left\lceil \frac{N}{k} \right\rceil - 1 \right) < k \left(\left(\frac{N}{k} + 1 \right) - 1 \right) = N,$$

where the inequality $\lceil N/k \rceil < \lceil N/k \rceil + 1$ has been used. This is a contradiction because there are a total of N objects. ◀

Example: Among 200 students in CS2214 there are at least $\lceil 200/12 \rceil = 17$ who were born in the same month.

The Generalized Pigeonhole Principle

Example: How many cards (N) must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

Solution: We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least $\lceil N/4 \rceil$ cards. At least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is

$$N = 2 \cdot 4 + 1 = 9.$$

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r -permutation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement $3,1,2$ is a permutation of S .
- The ordered arrangement $3,2$ is a 2-permutation of S .
- The number of r -permutations of a set with n elements is denoted by $P(n,r)$.
 - The 2-permutations of $S = \{1,2,3\}$ are
 $1,2; 1,3; 2,1; 2,3; 3,1; 3,2$. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r -permutations of a set with n distinct elements.

Proof: Use the product rule. The first element can be chosen in n ways. The second in $n - 1$ ways, and so on until there are $(n - (r - 1))$ ways to choose the last element. ◀

Note that $P(n, 0) = 1$ as there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n, r) = \frac{n!}{(n-r)!}$$

Solving Counting Problems by Counting Permutations

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970,200$$

Solving Counting Problems by Counting Permutations (*continued*)

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Solving Counting Problems by Counting Permutations (*continued*)

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Combinations

Definition: An *r -combination* of elements of a set is an unordered selection of r elements from the set. Thus, an r -combination is simply a subset of the set with r elements.

- The number of r -combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (We will see the notation again in the binomial theorem in Section 6.4.)

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c\}$ is a 2-combination from S . It is the same as $\{c, a\}$ since the order listed does not matter.

- $C(4, 2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of r -combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n, r) \cdot P(r, r)$.

procedure:
get ordered
arrangement
of r elements
from a set of n .

task 1:
get unordered
selection
of r elements
from a set of n .

task 2:
get ordered
arrangement
of r elements
from a set of r .

Therefore,

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$



Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$\begin{aligned} C(52, 5) &= \frac{52!}{5!47!} \\ &= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960 \end{aligned}$$

The different ways to select 47 cards from 52 is

$$C(52, 47) = \frac{52!}{47!5!} = C(52, 5) = 2,598,960.$$

This is a special case of a general result. →

Combinations


Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n, n - r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}.$$

Hence, $C(n, r) = C(n, n - r)$. 

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775 .$$

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?
 - To obtain x^3 , an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
 - To obtain x^2y , an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
 - To obtain xy^2 , an x must be chosen from one of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
 - To obtain y^3 , a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Proof: We use combinatorial reasoning. All terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for $j = 0, 1, 2, \dots, n$. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ x s from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$. ◀

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$.
By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

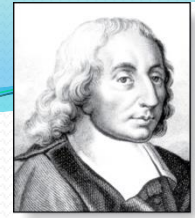
$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^n \binom{n}{k} = 2^n$.

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^n \binom{n}{k}. \quad \blacktriangleleft$$



Pascal's Identity

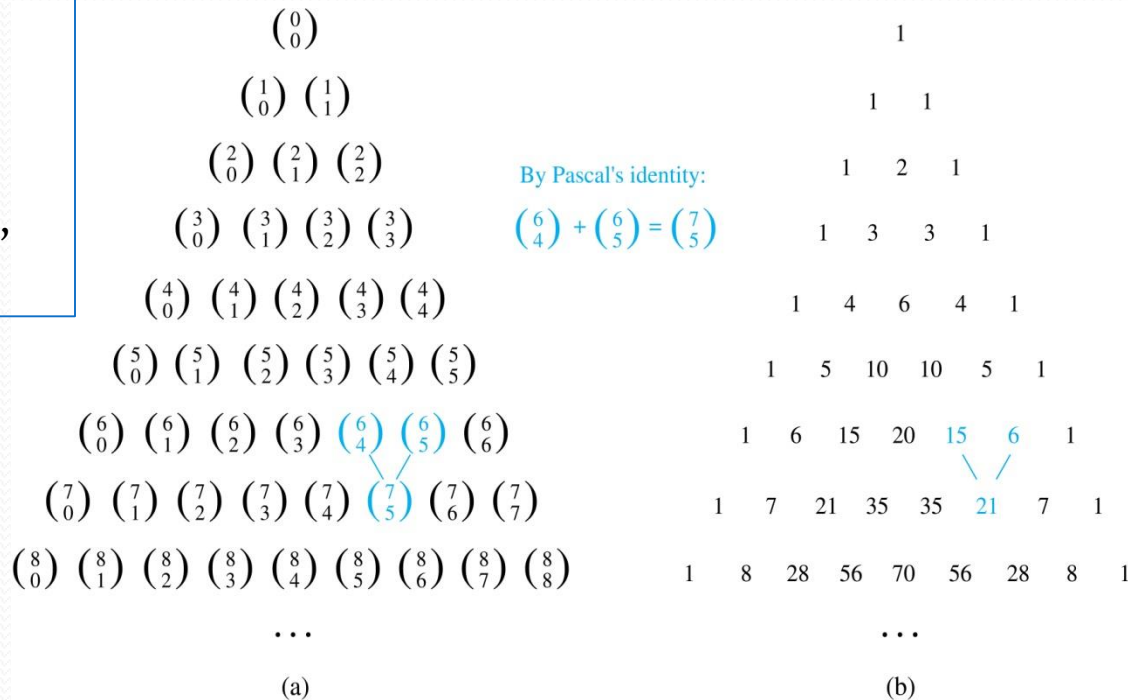
Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof: Exercise

Pascal's Triangle

The n th row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0, 1, \dots, n$.



By Pascal's identity, adding two adjacent binomial coefficients results in the binomial coefficient in the next row between these two coefficients.